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HITTING PARABOLAS AND EXPONENTIAL CURVES WITH DIFFUSION PROCESSES

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Summary
First passage time problems are considered for two-dimensional diffusion processes $(x_1(t), x_2(t))$ with infinitesimal variances proportional to $x_i^{k_i}$, for $i = 1, 2$. Conditions are obtained for the problems to be solvable by a particular case of the method of similarity solutions. Explicit solutions are presented.

Key Words: Hitting time; Kolmogorov equation; similarity solutions.

1 Introduction

Lefebvre and Labib (1996) considered the two-dimensional stochastic process defined by the system of stochastic differential equations

$$dx_i(t) = f_i(x_i(t))\,dt + dW_i(t), \tag{1.1}$$

for $i = 1, 2$, where $W_1(t)$ and $W_2(t)$ are independent one-dimensional standard Brownian motion processes. Let $M(x_1, x_2; a)$ be the moment generating function of the first passage time

$$T(x_1, x_2) := \inf\{t \geq 0 : (x_1(t), x_2(t)) \in D | x_i(0) = x_i, i = 1, 2\}, \tag{1.2}$$

where $D \subset I\!R^2$; that is,

$$M(x_1, x_2; a) := E[e^{-aT(x_1, x_2)}], \tag{1.3}$$

where we assume that a is a real, positive parameter. We have used the notation $M(x_1, x_2; a)$ to indicate that the last argument of the function M is not really a variable but rather a parameter.

In (1.2), D is a subset of $I\!R^2$ such that the random variable $T(x_1, x_2)$ is well-defined. For instance, the boundary ∂D of D could be a straight

line or a circle; however, D cannot be a single point in $\mathbb{R}^2$ because the probability of hitting a single point in $\mathbb{R}^2$ for a two-dimensional diffusion process is equal to zero.

Next, the moment generating function $M(x_1, x_2; a)$ satisfies the Kolmogorov backward equation

$$\sum_{i=1}^{2} \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_i^2} M(x_1, x_2; a) + f_i(x_i) \frac{\partial}{\partial x_i} M(x_1, x_2; a) \right\} = aM(x_1, x_2; a). \quad (1.4)$$

Lefebvre and Labib solved equation (1.4) by assuming that the function of two variables $M(x_1, x_2; a)$ can actually be written as a function of a *single* variable

$$z := \phi_1(x_1) + \phi_2(x_2); \quad (1.5)$$

that is,

$$M(x_1, x_2; a) = N(z; a). \quad (1.6)$$

Assuming also, without loss of generality, that $\phi_1(x_1)$ and $\phi_2(x_2)$ do not contain constant terms, they found that ϕ_i had to be such that

$$\phi_i(x_i) = \alpha x_i^2 + \beta_i x_i, \quad (1.7)$$

where α and β_i are constants.

Remarks. 1. By assuming that the function ϕ_i does not contain a constant term, we mean that $\phi_i(x_i)$ can be equal to x_i but not to $x_i + 1$, for instance.

2. There is no loss of generality in assuming that ϕ_i does not contain a constant term, because if we say that g is a function of $y := x + c$, then it is also a function of x. Conversely, if a function g depends on x, then we can also say that it depends on $x + c$ since we can replace x by $(x + c) - c$ everywhere x appears in $g(x)$.

Finally, Lefebvre and Labib gave a list of problems that could be solved explicitly by using this special case of the method of similarity solutions (see Dresner, 1983). Note that all these problems had already been solved by using other methods or were generalizations of such problems. Hence, the method of similarity solutions enabled them to retrieve known solutions.

In the present paper, we generalize the system (1.1) by considering the stochastic differential equations

$$dx_i(t) = f_i(x_i(t))\, dt + (c_i x_i^{k_i}(t))^{1/2} dW_i(t), \quad (1.8)$$

for $i = 1, 2$, where $c_i > 0$ and k_i are constants. Therefore, we assume that the infinitesimal variance of the process $x_i(t)$ is proportional to $x_i^{k_i}(t)$, which must hence be non-negative. We consider the case when k_i is non-negative.

The moment generating function of the random variable $T(x_1, x_2)$ (see (1.3)) then satisfies the Kolmogorov backward equation

$$\sum_{i=1}^{2} \left\{ \frac{1}{2} c_i x_i^{k_i} \frac{\partial^2}{\partial x_i^2} M(x_1, x_2; a) + f_i(x_i) \frac{\partial}{\partial x_i} M(x_1, x_2; a) \right\} = aM(x_1, x_2; a). \tag{1.9}$$

This equation is valid in D^c, the complement of the stopping region D in $\mathbb{R}^2$. Moreover, we have the boundary condition

$$M(x_1, x_2; a) = 1 \quad \forall (x_1, x_2) \in \partial D, \tag{1.10}$$

where ∂D is the boundary of the set D. Finally, if we assume that the probability $P[T(x_1, x_2) < \infty]$ is strictly positive, then the function M must satisfy the inequalities

$$0 < M(x_1, x_2; a) < 1 \quad \forall (x_1, x_2) \in D^c. \tag{1.11}$$

Indeed, if $(x_1, x_2) \in D^c$ and $P[T(x_1, x_2) < \infty] > 0$, then the exponential $e^{-aT(x_1,x_2)}$ is always in the interval $(0, 1)$, and so is its expected value.

Again, our aim is to solve explicitly equation (1.9) by assuming that $M(x_1, x_2; a) = N(z; a)$, with $z := \phi_1(x_1) + \phi_2(x_2)$. As in Lefebvre and Labib (1996), we postulate two hypotheses concerning the boundary ∂D in order to obtain real, well-defined two-dimensional problems:

H1: the boundary ∂D depends explicitly on both x_1 and x_2; that is, ∂D could be the straight line defined by $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = r\}$, but not the straight line defined by $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = r\}$.

H2: the boundary ∂D consists of a single continuous curve. Therefore, ∂D cannot be a finite or countably infinite set of points.

Note that the hypothesis H1 implies that $\phi_i(x_i)$ is not constant. Indeed, the time taken to hit the straight line defined by $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = r\}$, for instance, must depend on both x_1 and x_2. Moreover, for simplicity we assume, as above, that ϕ_1 and ϕ_2 do not contain constant terms (see the preceding remark). Note also that we could consider what is termed in the literature "two-boundary problems", rather than problems with a boundary ∂D consisting of a continuous curve. That is, the continuation region D^c could be the region between two parallel lines or between two concentric circles, etc. Then the boundary ∂D would be the union of the two straight lines or the two circles, etc.

The stochastic processes defined by the stochastic differential equation (1.8) have many applications. In particular, stochastic processes of this form have been used in mathematical finance. For instance, the process defined by

$$dx(t) = (a + bx(t))dt + \sigma x^{1/2}(t) dW(t) \tag{1.12}$$

is called the Cox-Ingersoll-Ross (CIR) model (see Maghsoodi, 1996, for example). It is also called the square root process. It has been used in

developing valuation models for interest-rate-sensitive contingent claims. This model implies that the conditional volatility of changes in $x(t)$ is proportional to $x(t)$.

Another particular case of importance is the one when

$$dx(t) = adt + \sigma dW(t), \tag{1.13}$$

which is simply a Brownian motion with drift a. It has been used to derive a model of discount bond prices. In this case, the volatility of changes in the variable $x(t)$ is assumed constant. The stochastic differential equation (1.13) can be generalized to

$$dx(t) = (a + bx(t))dt + \sigma dW(t), \tag{1.14}$$

and this generalization has been used to derive an equilibrium model of discount bond prices. Notice that if $a = 0$ (and $b < 0$), then $x(t)$ is an Ornstein-Uhlenbeck process.

Finally, the Bessel process is also a special case of (1.8) since it corresponds to the choice

$$f_i(x_i(t)) = \frac{\alpha - 1}{2x_i(t)} \tag{1.15}$$

and

$$c_i x_i^{k_i}(t) \equiv \sigma^2. \tag{1.16}$$

In mathematical finance, Bessel processes have been used to evaluate Asian options (see Geman and Yor (1993) and also Chesney *et al.* (1993)).

Another application of the stochastic processes defined by (1.8) is in genetics. Indeed, for instance the author (Lefebvre, 1998) has considered first passage problems for the diffusion process defined by

$$dx(t) = \mu dt + (2x(t))^{1/2} dW(t), \tag{1.17}$$

which can be used to study the fluctuations of the number of A-type individuals in a population consisting of two types (A and a) of individuals, when they are of the order $\sqrt{N}$, where N is the constant size of the population.

Karlin and Taylor (1981, pp. 176-191) give a list of diffusion processes that can serve as approximations in population genetic models. In addition to the model above, this list also includes the Ornstein-Uhlenbeck process.

In Section 2, we obtain the formulae that correspond to (1.7); that is, we compute the admissible functions ϕ_i in the present problem. Furthermore, we find what functions f_i can be used in equation (1.8), once the admissible functions ϕ_i are known. Next, in Section 3, particular problems are considered and solved explicitly (by using the method of similarity solutions) for diffusion processes of the type described above. These first passage problems have obvious applications in both mathematical finance and population genetics, among many other possible applications. Finally, some concluding remarks are made in Section 4.

2 Theoretical results

Using the representation $M(x_1, x_2; a) = N(\phi_1(x_1) + \phi_2(x_2); a) := N(z; a)$, we find that equation (1.9) is transformed to

$$\sum_{i=1}^{2} \left\{ \frac{1}{2} c_i x_i^{k_i} \phi_i'^2(x_i) N''(z; a) + \left[\frac{1}{2} c_i x_i^{k_i} \phi_i''(x_i) + f_i(x_i) \phi_i'(x_i) \right] N'(z; a) \right\} = aN, \tag{2.1}$$

where $N'(z; a) := \frac{d}{dz} N(z; a)$ and $N''(z; a) := \frac{d^2}{dz^2} N(z; a)$.

Remark. Remember that a is a parameter, so that N is a function of a single variable. We could have treated a as a variable and written $N(z; a)$ as $N(z, a)$; then, we would write $\frac{\partial}{\partial z}$. However, it is more customary to make use of the former notation.

We will obtain the form that the function ϕ_i can take. But first we need the following lemma.

Lemma 2.1 *The function $N(z; a)$ cannot be written as*

$$N(z; a) = F(z)G(a) + K(a) \quad \textit{for any } a > 0. \tag{2.2}$$

Proof. The proof is essentially the same as in Lefebvre and Labib (1996) and relies on the fact that the function $G(a)$ must be different from zero for all $a > 0$, for any point $(x_1, x_2) \in D^c$. Otherwise the function $N(z; a)$ would not depend on z for some point $(x_1, x_2) \in D^c$, and thus we would deduce from (2.1) that $N(z; a) = 0$, which is not possible. Similarly if the function $F(z)$ is constant. □

The main theoretical result of this paper now follows.

Proposition 2.1 *Let $(x_1(t), x_2(t))$ be the two dimensional stochastic process defined by (1.8) and*

$$T(x_1, x_2) := \inf\{t \geq 0 : \phi_1(x_1(t)) + \phi_2(x_2(t)) = K | x_i(0) = x_i, i = 1, 2\}, \tag{2.3}$$

where K is a constant and we assume that the second derivative $\phi_i''(x_i)$ exists and the derivative $\phi_i'(x_i) \neq 0$ for $i = 1, 2$, $\forall (x_1, x_2) \in D^c$. That is, the subset D of $\mathbb{R}^2$ is defined by

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : \phi_1(x_1) + \phi_2(x_2) < K \textit{ (respectively } > K)\}$$

if $\phi_1(x_1(0)) + \phi_2(x_2(0)) > K$ (resp. $< K$).

Let $M(x_1, x_2; a)$ be the moment generating function (defined in (1.3)) of the first passage time $T(x_1, x_2)$. Then a necessary condition for the representation

$$M(x_1, x_2; a) = N(\phi_1(x_1) + \phi_2(x_2); a) := N(z; a) \tag{2.4}$$

to be valid, all of the functions $N(z;a), N'(z;a)$ and $N''(z;a)$ being differentiable with respect to the parameter a, is that

$$\phi_i(x_i) = \frac{b_i}{(2-k_i)^2} x_i^{2-k_i} + \frac{d_i}{2-k_i} x_i^{1-(k_i/2)} \quad \text{if } k_i \neq 2 \tag{2.5}$$

and

$$\phi_i(x_i) = \frac{b_i}{4} \ln^2(x_i) + d_i \ln(x_i) \quad \text{if } k_i = 2, \tag{2.6}$$

where b_i and d_i are constants, for $i = 1, 2$. Furthermore, we must have

$$c_1 b_1 = c_2 b_2. \tag{2.7}$$

Proof. See Lefebvre and Labib (1996). Since $T(x_1, x_2)$ (which we could write as $T(x_1, x_2; K)$) must increase if the stochastic process $\phi_1(x_1(t)) + \phi_2(x_2(t))$ moves away from K, we deduce that $N(z;a)$ is strictly decreasing as z increases (if $z \geq K$) or as z decreases (if $z \leq K$). It follows that $N'(z;a) < 0 \ \forall z > K$ (or $z < K$). Hence we can rewrite equation (2.1) as

$$\sum_{i=1}^{2} \left\{ \frac{1}{2} c_i x_i^{k_i} \phi_i'^2(x_i) H(z;a) + \left(\frac{1}{2} c_i x_i^{k_i} \phi_i''(x_i) + f_i(x_i) \phi_i'(x_i) \right) \right\} = L(z;a), \tag{2.8}$$

where

$$H(z;a) := \frac{N''(z;a)}{N'(z;a)} \quad \text{and} \quad L(z;a) := \frac{aN(z;a)}{N'(z;a)}. \tag{2.9}$$

Next, differentiating both sides of equation (2.8) with respect to a, we obtain that

$$\frac{1}{2} \sum_{i=1}^{2} c_i x_i^{k_i} \phi_i'^2(x_i) \frac{\partial}{\partial a} H(z;a) = \frac{\partial}{\partial a} L(z;a). \tag{2.10}$$

We must have $\frac{\partial}{\partial a} H(z;a) \neq 0$ for all $a > 0$. Indeed, if $\frac{\partial}{\partial a} H(z;a) = 0$ for some $a > 0$, then we can show that

$$N(z;a) = F(z)G(a) + K(a) \tag{2.11}$$

for this a, which contradicts Lemma 2.1. Therefore we deduce that

$$\frac{1}{2} \sum_{i=1}^{2} c_i x_i^{k_i} \phi_i'^2(x_i) = \sigma(z), \tag{2.12}$$

where the function $\sigma(z)$ can be a constant.

Now, equation (2.12) implies that

$$\frac{\partial^2}{\partial x_1 \partial x_2}\sigma(z) = 0 \Longleftrightarrow \sigma''(z)\phi_1'(x_1)\phi_2'(x_2) = 0 \quad \forall (x_1, x_2) \in D^c. \tag{2.13}$$

Since we cannot have $\phi_i'(x_i) \equiv 0$ (because $\phi_i(x_i)$ cannot be a constant), we set $\sigma''(z) \equiv 0$, so that

$$\sigma(z) = \sigma_1 z + \sigma_0 \quad (\sigma_1 \text{ and } \sigma_0 \text{ constants}). \tag{2.14}$$

That is,

$$\frac{1}{2}\sum_{i=1}^{2} c_i x_i^{k_i}\phi_i'^2(x_i) = \sigma_1(\phi_1(x_1) + \phi_2(x_2)) + \sigma_0. \tag{2.15}$$

It follows that

$$\frac{1}{2}c_i x_i^{k_i}\phi_i'^2(x_i) = \sigma_1\phi_i(x_i) + \text{constant} \quad \text{for } i = 1, 2. \tag{2.16}$$

Remembering that we assumed that the functions $\phi_i(\cdot)$ do not contain constant terms, we may write that

$$\phi_i(x_i) = \frac{2\sigma_1}{c_i(2-k_i)^2}x_i^{2-k_i} + \frac{d_i}{2-k_i}x_i^{1-(k_i/2)} \quad \text{if } k_i \neq 2 \tag{2.17}$$

and

$$\phi_i(x_i) = \frac{\sigma_1}{2c_i}\ln^2(x_i) + d_i\ln(x_i) \quad \text{if } k_i = 2, \tag{2.18}$$

where d_i is a constant, for $i = 1, 2$. □

Remark. With $c_1 = c_2 = 1$ and $k_1 = k_2 = 0$, as in Lefebvre and Labib (1996), we retrieve the formula

$$\phi_i(x_i) = \alpha x_i^2 + \beta_i x_i, \tag{2.19}$$

where α and β_i are constants.

The next result gives us the functions f_i that can be used in (1.8).

Corollary 2.1 *For our special case of the method of similarity solutions to enable us to solve the problems considered in this note, a second necessary condition is that the function f_i that appears in (1.8) be such that*

$$f_i(x_i) = \frac{1}{\phi_i'(x_i)}\left(\mu_1\phi_i(x_i) - \frac{1}{2}c_i x_i^{k_i}\phi_i''(x_i) + e_i\right), \tag{2.20}$$

where μ_1 and e_i are constants, and we assume that $\phi_i'(x_i) \neq 0$ $\forall (x_1, x_2) \in D^c$, for $i = 1, 2$.

Proof. Using equation (2.12), equation (2.1) becomes

$$\sigma(z)N''(z;a) + \sum_{i=1}^{2}\left\{\frac{1}{2}c_i x_i^{k_i}\phi_i''(x_i) + f_i(x_i)\phi_i'(x_i)\right\}N'(z;a) = aN(z;a). \tag{2.21}$$

Because $N'(z;a) < 0\ \forall z > K$ (or $z < K$), we deduce that

$$\sum_{i=1}^{2}\left\{\frac{1}{2}c_i x_i^{k_i}\phi_i''(x_i) + f_i(x_i)\phi_i'(x_i)\right\} = \mu(z). \tag{2.22}$$

As in the proof of Proposition 2.1, we have

$$\frac{\partial^2}{\partial x_1 \partial x_2}\mu(z) = \mu''(z)\phi_1'(x_1)\phi_2'(x_2) = 0 \quad \forall (x_1, x_2) \in D^c, \tag{2.23}$$

so that $\mu''(z) \equiv 0$; that is,

$$\mu(z) = \mu_1 z + \mu_0 = \mu_1(\phi_1(x_1) + \phi_2(x_2)) + \mu_0 \quad (\mu_1 \text{ and } \mu_0 \text{ constants}). \tag{2.24}$$

Substituting this into equation (2.22), we deduce that $f_i(x_i)$ is indeed such as in (2.20). □

Remark. Again, Lefebvre and Labib (1996) took $c_1 = c_2 = 1$ and $k_1 = k_2 = 0$. Using these values in (2.20), together with the formula $\phi_i(x_i) = \alpha x_i^2 + \beta_i x_i$, we find that

$$f_i(x_i) = \frac{1}{2\alpha x_i^2 + \beta_i}\Big(\mu_1(\alpha x_i^2 + \beta_i x_i) + \epsilon_i\Big), \tag{2.25}$$

where ϵ_i is a constant, for $i = 1, 2$.

Now, in Lefebvre and Labib (1996) it is stated that we must have either $\alpha \neq 0$ and $\beta_1 = \beta_2 = 0$ or $\alpha = 0$ and $\beta_1\beta_2 \neq 0$. This implies that the function $f_i(x_i)$ must be of the form $f_i(x_i) = m_i$, $m_i x_i$, or m_i/x_i, where m_i is a constant. These results are in fact valid if we assume that $f_i(x_i)$ is proportional to a power of x_i, which is certainly the most important case in practice. This hypothesis was implicit in Lefebvre and Labib (1996). However, we see here that $f_i(x_i)$ is not necessarily of this form. There are some particular cases of the formula (2.25) which lead to interesting problems. We present such a particular case in Section 3. We also present two other problems which are special cases of the general problem considered in this note and that are solved explicitly. As far as we know, these three problems have no known solutions yet.

3 Particular cases

I) The first particular case that we consider is the one where

$$dx_i(t) = -\beta x_i(t)\,dt + \frac{\alpha_i - 1}{2x_i(t)}\,dt + dW_i(t) \tag{3.1}$$

for $i = 1, 2$, where β (> 0) and α_i (≥ 0) are constants. Note that if we take $\beta = 0$ in (3.1), then $x_i(t)$ is a Bessel process (see Karlin and Taylor, 1981, p. 175), whereas if we choose $\alpha_1 = \alpha_2 = 1$, then $x_i(t)$ is an Ornstein-Uhlenbeck process. The process $(x_1(t), x_2(t))$ defined by (3.1) is thus a generalization of two cases treated in Lefebvre and Labib (1996) and which have known solutions. We take

$$T(x_1, x_2) := \inf\{t \geq 0 : x_1^2(t) + x_2^2(t) = r^2 | x_i(0) = x_i, i = 1, 2\}. \tag{3.2}$$

That is, $\phi_i(x_i(t)) = x_i^2(t)$ and $K = r^2$. Furthermore, we assume that $x_1^2 + x_2^2 \leq r^2$. Therefore, $T(x_1, x_2)$ is the time it takes the two-dimensional stochastic process $(x_1(t), x_2(t))$ to hit the circle defined by

$$\partial D = \{(x_1, x_2) \in I\!R^2 : x_1^2 + x_2^2 = r^2\}, \tag{3.3}$$

starting inside the circle.

This problem could be interpreted as that of finding the distribution of the time it takes the sum of the (squared) sizes of two populations, or the sum of the (squared) prices of bonds, to reach a certain level.

The moment generating function $M(x_1, x_2; a) := E[e^{-aT(x_1,x_2)}]$ satisfies the Kolmogorov backward equation

$$\sum_{i=1}^{2} \left\{ \frac{1}{2}\frac{\partial^2}{\partial x_i^2} M(x_1, x_2; a) + \left(-\beta x_i + \frac{\alpha_i - 1}{2x_i} \right) \frac{\partial}{\partial x_i} M(x_1, x_2; a) \right\} = aM. \tag{3.4}$$

We try a solution of the form $M(x_1, x_2; a) = N(z; a)$, with $z := x_1^2 + x_2^2$. Then equation (3.4) becomes

$$2zN''(z; a) + (-2\beta z + (\alpha_1 + \alpha_2))N'(z; a) = aN(z; a). \tag{3.5}$$

We will solve explicitly the previous equation in the special case when $\beta = 1$ and $\alpha_1 = \alpha_2 = 3/2$. With these choices, we have the following lemma.

Lemma 3.1 *The stochastic process $x(t)$ defined by the stochastic differential equation*

$$dx(t) = -x(t)\,dt + \frac{1}{4x(t)}\,dt + dW(t) \tag{3.6}$$

has an inaccessible boundary at the origin.

Proof. This result can be proved by using the formulae in Kannan (1979, p. 279), for instance. □

Remark. Lemma 3.1 implies that if $x_1 > 0$ and $x_2 > 0$, then the two-dimensional process $(x_1(t), x_2(t))$ cannot leave the first quadrant.

The equation that we must solve is

$$2zN''(z;a) + (3-2z)N'(z;a) = aN(z;a). \tag{3.7}$$

Its general solution is found to be

$$N(z;a) = r_1\, U(a/2, 3/2, z) + r_2\, {}_1F_1(a/2, 3/2, z), \tag{3.8}$$

where r_i is a constant for $i = 1, 2$, and $U(\cdot,\cdot,\cdot)$ and ${}_1F_1(\cdot,\cdot,\cdot)$ are confluent hypergeometric functions (see Abramowitz and Stegun, 1965, p. 504).

Now, equation (3.8) contains two arbitrary constants and we have only one boundary condition, namely

$$N(r^2;a) = 1. \tag{3.9}$$

But, using the fact that the function $U(a/2, 3/2, z)$ diverges as z decreases to zero (see Abramowitz and Stegun, 1965, p. 508), we deduce that we must set r_1 equal to zero in (3.8); otherwise, the condition $0 < M(x_1, x_2; a) < 1$ $\forall (x_1, x_2) \in D^c$ (see (1.11)) will not be fulfilled. Then, making use of the boundary condition (3.9), we obtain that

$$N(z;a) = \frac{{}_1F_1(a/2, 3/2, z)}{{}_1F_1(a/2, 3/2, r^2)} \tag{3.10}$$

for $0 < z \leq r^2$.

Remark. The function ${}_1F_1(a, b, z)$ is sometimes denoted by $M(a, b, z)$, in Abramowitz and Stegun (1965, p. 504), in particular. However, here we have already used the (standard) notation $M(x_1, x_2; a)$ in (1.3) for the moment generating function of the random variable $T(x_1, x_2)$. Therefore, it is better to use the notation ${}_1F_1(a, b, z)$ to avoid possible confusion.

Proposition 3.1 *The moment generating function of the random variable $T(x_1, x_2)$ defined in (3.2) is given by*

$$M(x_1, x_2; a) = \frac{{}_1F_1(a/2, 3/2, x_1^2 + x_2^2)}{{}_1F_1(a/2, 3/2, r^2)} \tag{3.11}$$

for all couples (x_1, x_2) such that $x_i > 0$ for $i = 1, 2$ and $x_1^2 + x_2^2 \leq r^2$.

Proof. All that remains to do is to make sure that the solution that we have obtained, by using the method of similarity solutions as above, is really the one that we were looking for. Now, we have (see Abramowitz and Stegun, 1965, p. 504)

$$P[T(x_1, x_2) < \infty] = \lim_{a \downarrow 0} M(x_1, x_2; a) = 1. \tag{3.12}$$

It follows that there is a *unique* solution to equation (3.4), subject to the appropriate boundary condition. Hence, the solution that we have obtained is indeed the moment generating function of the random variable $T(x_1, x_2)$. □

Another application of the problem that we have solved is in statistical quality control. Indeed, assume that a company is producing objects that possess two characteristics, $x_1(t)$ and $x_2(t)$, that vary in time (approximately) according to the stochastic differential equations (3.1). Assume also that, ideally, the values of $x_1(t)$ and $x_2(t)$ should be zero. Then the random variable $T(x_1, x_2)$ represents the time that elapses before the joint deviation from the optimal value of $(x_1(t), x_2(t))$ (as measured by the distance from $(x_1(t), x_2(t))$ to the origin) reaches a given threshold r^2, a level at which it is necessary to stop the production and make some adjustments. So, we would be interested in knowing how long will the production continue before it must be stopped, when the current value of $x_i(t)$ is x_i for $i = 1, 2$.

As mentioned previously, this first example is an (original) extension of the work presented in Lefebvre and Labib (1996), obtained by assuming that the functions $f_i(x_i)$ are not necessarily proportional to powers of x_i. This example is less important than the ones listed in Lefebvre and Labib (1996), but it can certainly be used in some applications. The second and third examples that we present are, on the other hand, completely new first passage time problems which involve the time it takes to hit a parabola and an exponential curve, respectively.

II) We consider the stochastic process $(x_1(t), x_2(t))$ defined by the system of stochastic differential equations

$$dx_1(t) = (\frac{1}{2})^{1/2}\, dW_1(t), \tag{3.13}$$

$$dx_2(t) = \mu\, dt + (2x_2(t))^{1/2}\, dW_2(t), \tag{3.14}$$

where $\mu \geq 1$ is a constant. Note that the origin is an inaccessible boundary for the $x_2(t)$ process (see Lefebvre, 1998), whereas $x_1(t)$ is a Brownian motion process with infinitesimal mean 0 and infinitesimal variance 1/2. We let

$$T(x_1, x_2) := \inf\{t \geq 0 : x_1^2(t) + x_2(t) = r\ (> 0) | x_i(0) = x_i, i = 1, 2\}. \tag{3.15}$$

That is, this time $\phi_1(x_1(t)) = x_1^2(t)$, $\phi_2(x_2(t)) = x_2(t)$ and $K = r$. Furthermore, we assume that $x_2 > 0$ and $x_1^2 + x_2 \leq r$.

As in the case of the first example presented in this section, this problem has an obvious interpretation in mathematical finance and in population genetics since $x_1(t)$ and $x_2(t)$ can both represent the size of a population and $x_2(t)$ is a special case of the CIR model used in mathematical finance.

Another application of the model is in hydrology. Indeed, diffusion processes have recently been used with success to model the flow of rivers and catchment basins (see Labib *et al.*, 2000). Assume, here, that the flow of a certain river at time t is obtained by adding the quantities $x_1^2(t)$ and $x_2(t)$ as defined above. For instance, $x_1^2(t)$ and $x_2(t)$ can be the contribution to the flow of the river from its two tributaries. Then, $T(x_1, x_2)$ is the time it takes the river flow to reach a level at which there could be a significant risk of flooding, so that actions must be taken to prevent it.

The partial differential equation that we have to solve is now

$$\frac{1}{4}\frac{\partial^2}{\partial x_1^2}M(x_1, x_2; a) + x_2 \frac{\partial^2}{\partial x_2^2}M(x_1, x_2; a) + \mu \frac{\partial}{\partial x_2}M(x_1, x_2; a) = aM. \quad (3.16)$$

We write that $M(x_1, x_2; a) = N(z; a)$, with $z := x_1^2 + x_2$. Then, we find that

$$zN''(z; a) + (\mu + 1/2)N'(z; a) = aN(z; a), \quad (3.17)$$

the general solution of which being

$$N(z; a) = z^{1/2(1/2-\mu)}\Big(r_1 I_{1/2-\mu}(2\sqrt{az}) + r_2 K_{1/2-\mu}(2\sqrt{az})\Big), \quad (3.18)$$

where $I_\nu(\cdot)$ and $K_\nu(\cdot)$ are modified Bessel functions (see Abramowitz and Stegun, 1965, p. 374).

Let us choose $\mu = 3/2$. Then, we have

$$N(z; a) = z^{-1/2}\Big(r_1 I_{-1}(2\sqrt{az}) + r_2 K_{-1}(2\sqrt{az})\Big), \quad (3.19)$$

which can be rewritten as

$$N(z; a) = z^{-1/2}\Big(r_1 I_1(2\sqrt{az}) + r_2 K_1(2\sqrt{az})\Big) \quad (3.20)$$

(see Abramowitz and Stegun, 1965, p. 375).

As in the previous example, the fact that $K_1(z)$ diverges as $z \downarrow 0$ implies that we must set r_2 equal to zero (otherwise (1.11) will be violated), which, together with the boundary condition

$$N(r; a) = 1, \quad (3.21)$$

yields that

$$N(z;a) = \left(\frac{r}{z}\right)^{1/2} \frac{I_1(2\sqrt{az})}{I_1(2\sqrt{ar})} \quad \text{for } 0 < z \leq r. \tag{3.22}$$

Note that (see Abramowitz and Stegun, 1965, p. 375)

$$P[T(x_1,x_2) < \infty] = \lim_{a \downarrow 0} N(z;a) = \lim_{a \downarrow 0} \left(\frac{r}{z}\right)^{1/2} \frac{\sqrt{az}}{\sqrt{ar}} = 1, \tag{3.23}$$

so that the solution to our problem is *unique*. Hence, we have now proved the following proposition.

Proposition 3.2 *The moment generating function of the random variable* $T(x_1,x_2)$ *defined in (3.15) is given by*

$$M(x_1,x_2;a) = \left(\frac{r}{x_1^2+x_2}\right) \frac{I_1\left(2\sqrt{a(x_1^2+x_2)}\right)}{I_1(2\sqrt{ar})} \tag{3.24}$$

for $x_2 > 0, x_1^2 + x_2 \leq r$.

Note that the continuation region in this problem (and in example I as well) is a *bounded* subset of $\mathbb{R}^2$. It is defined by the area between the x_1-axis and the parabola $x_1^2 + x_2 = r$ (> 0). Note also that if we define $y_1(t) = x_1^2(t)$, then $y_1(t)$ is a diffusion process with infinitesimal parameters $1/2$ and $2y_1(t)$ (see Karlin and Taylor, 1981, p. 173). Hence, $x_1^2(t)$ is actually the special case of the $x_2(t)$ process when $\mu = 1/2$. Therefore, $x_1^2(t)$ and $x_2(t)$ are processes of the same type, that is, processes with similar infinitesimal parameters, in the example above.

III) Finally, we consider the stochastic process $(x_1(t), x_2(t))$ defined by

$$dx_1(t) = 2x_1^3(t)\,dt + (2x_1^4(t))^{1/2}\,dW_1(t), \tag{3.25}$$

$$dx_2(t) = x_2(t)\,dt + (2x_2^2(t))^{1/2}\,dW_2(t). \tag{3.26}$$

In this case, the process $x_2(t)$ is a geometric Brownian motion (see Karlin and Taylor, 1981, p. 175). Because this process can be represented as the exponential of a Brownian motion process, it has an inaccessible boundary at the origin.

Next, we choose

$$T(x_1,x_2) := \inf\{t \geq 0 : x_2(t) = \exp[r - x_1^{-1}(t)] | x_i(0) = x_i, i = 1,2\}. \tag{3.27}$$

Thus, we take $\phi_1(x_1(t)) = 1/x_1(t)$, $\phi_2(x_2(t)) = \ln(x_2(t))$ and $K = r$. We assume that $x_i > 0$ for $i = 1, 2$ and that $\ln(x_2) + x_1^{-1} \leq r$. It follows that the continuation region D^c is defined by the area, in the first quadrant, between the x-axis and the exponential curve $x_2 = \exp[r - x_1^{-1}]$. This example is a purely mathematical one; however, it is interesting because the boundary of the stopping region is an exponential curve.

To solve the Kolmogorov backward equation

$$x_1^4 \frac{\partial^2}{\partial x_1^2} M + x_2^2 \frac{\partial^2}{\partial x_2^2} M + 2x_1^3 \frac{\partial}{\partial x_1} M + x_2 \frac{\partial}{\partial x_2} M = aM, \tag{3.28}$$

where $M = M(x_1, x_2; a)$, we write that $M(x_1, x_2; a) = N(z; a)$, with

$$z := \ln(x_2) + x_1^{-1}. \tag{3.29}$$

We find that equation (3.28) is transformed to the very simple ordinary differential equation

$$2N''(z; a) = aN(z; a). \tag{3.30}$$

As is well known, this last equation has the general solution

$$N(z; a) = r_1 \exp\left[\sqrt{a/2}\, z\right] + r_2 \exp\left[-\sqrt{a/2}\, z\right]. \tag{3.31}$$

Now, it is easy to check that the variable $z = \ln(x_2) + x_1^{-1}$ varies from $-\infty$ to r in the continuation region D^c. It follows that we must set r_2 equal to zero in (3.31). Finally, because $N(r; a) = 1$, we obtain that

$$N(z; a) = \exp\left[\sqrt{a/2}(z - r)\right] \quad \text{for } z \leq r. \tag{3.32}$$

It is also a simple matter to check that

$$P[T(x_1, x_2) < \infty] = \lim_{a \downarrow 0} N(z; a) = 1, \tag{3.33}$$

which guarantees the *uniqueness* of the solution (although the continuation region D^c is *not* bounded in this example) and allows us to state the following proposition.

Proposition 3.3 *The moment generating function of the random variable $T(x_1, x_2)$ defined in (3.27) is given by*

$$M(x_1, x_2; a) = \exp\left[\sqrt{a/2}(x_1^{-1} + \ln(x_2) - r)\right] \tag{3.34}$$

for $x_i > 0$, $i = 1, 2$, and $x_1^{-1} + \ln(x_2) \leq r$.

As mentioned above, this third particular case was purely mathematical; it was presented to illustrate the usefulness of the results proved in Section 2. However, there are many other problems that can be solved explicitly by using Proposition 2.1 and Corollary 2.1.

4 Conclusion

In this note, we have generalized the work presented in Lefebvre and Labib (1996) by considering the case of not necessarily constant infinitesimal variances. This generalization is important because, as we have seen, the results proved in the present note enabled us to obtain explicit solutions to *new* first passage time problems in two dimensions, whereas Lefebvre and Labib only retrieve known solutions to classic problems. We also extended the theoretical results contained in Lefebvre and Labib (1996) by accepting that the function f_i in (1.8) can be any function for which $x_i(t)$ is a well-defined stochastic process, instead of assuming that $f_i(x_i)$ is proportional to a power of x_i (as should have been explicitly stated in Lefebvre and Labib, 1996). In Corollary 2.1, we obtained the functions f_i that are admissible in our problem. In Section 3, we presented three particular problems that we were able to solve explicitly. Example II seems particularly interesting to us, because it gives the solution to a significant problem, namely that of finding the time it takes a relatively important two-dimensional diffusion process to hit a parabola, which is a "natural" boundary to consider. Furthermore, this example is only one of numerous problems that can be solved by using the method of similarity solutions.

Finally, as mentioned in Lefebvre and Labib (1996), a possible sequel to the present work could be to try to apply the method of similarity solutions to problems in other coordinate systems, for example in polar coordinates. Also, other special cases of this method than the one used here could be tried.

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