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ROBUSTNESS OF THE SEQUENTIAL TESTING PROCEDURES FOR THE GENERALIZED LIFE DISTRIBUTIONS

Ajit Chaturvedi[1], **Neeraj Tiwari**[2] and **Sanjeev K. Tomer**[3]

[1] *Department of Statistics, University of Delhi, Delhi - 110007, India. Email: dustats@del3.vsnl.net.in*

[2] *Department of Statistics, Kumaun University, Almora - 263601, India*

[3] *Department of Statistics, Allahabad University, Allahabad - 211002, India. Email: anoopc@sancharnet.in*

Summary

Sequential testing procedures are developed for the parameters of generalized life distributions. Robustness of the testing procedures is studied, when the distribution under consideration has undergone a change. In order to apply Newton-Raphson method for the purpose of plotting the operating characteristic (OC) and average sample number (ASN) functions, a method of choosing the initial values is provided.

Key words: Generalized life distributions; OC and ASN functions; robustness; sequential probability ratio tests.

1 Introduction

The pioneering work in the area of sequential tests of statistical hypotheses is due to Wald (1947), who developed sequential probability ratio test (SPRT) for testing a simple hypothesis against the simple alternative. As measures of the performance of SPRT, Wald obtained the expressions for the OC and ASN functions. The robustness of the SPRT, when the distribution under consideration has undergone a change, has been studied by various authors while dealing with different probabilistic models useful in reliability/survival analysis. Epstein and Sobel (1955) developed SPRT for testing simple null hypothesis against simple alternative, for the scale parameter of an exponential distribution. Harter and Moore (1976) conducted Monte Carlo study to investigate the robustness of exponential SPRT when the underlying distribution is a Weibull with shape parameter other than one. Montagne and Singpurwalla (1985) generalized the results of Harter and Moore (1976) from Weibull to a class of distributions having an increasing (decreasing) failure rate. They obtained inequalities for OC and ASN functions in order to demonstrate the robustness. Phatarfod (1971) developed SPRT for testing composite hypothesis for the shape parameter of the gamma distribution. Recently, Chaturvedi, Kumar and Kumar (1998) proposed SPRTS for testing simple hypotheses (versus simple alternatives) for the parameters θ, a and b, when the random variable (rv) X follows the family of life-testing models presented by the probability density function (pdf)

$$f(x;\theta,a,b,c)=\frac{cx^{ac-1}\exp(-x^c/\theta^b)}{\theta^{ab}\,\Gamma(a)};\quad (x,\theta,a,b,c)>0. \tag{1.1}$$

The robustness of the SPRTS was studied in respect of OC and ASN functions, when the distribution under consideration had undergone a change.

In the present paper, we consider a more general form of the probabilistic model. Let the rv X follows the distribution presented by the probabilistic model

$$f(x;\delta,\theta)=\frac{g^{\delta-1}g'(x)}{\theta^\delta\Gamma(\delta)}\exp\left(-\frac{g(x)}{\theta}\right);\ x>a,\ (g(x),\theta,\delta)>0, \tag{1.2}$$

where 'a' is known and δ and θ are the parameters. Here, $g(x)$ is real-valued, strictly increasing function of x with $g(a)=0$ and $g'(x)$ stands for the derivative of $g(x)$.

We call the model (1.2) as the 'generalized life distributions'. It covers the following life distributions as specific cases:

(i) For $g(x)=x$, $a=0$ and $\delta=1$, we get the one-parameter exponential distribution [see Johnson and Kotz (1970, p. 166)].

(ii) For $g(x)=x$ and $a=0$, (1.2) becomes the gamma distribution and for δ taking integer values, it is known as Erlang distribution [see Johnson and Kotz (1970, p. 166)].

(iii) For $g(x)=x^p$ and $a=0$, (1.2) gives the generalized gamma distribution [see Johnson and Kotz (1970, p. 197)].

(iv) For $g(x)=x^p$, $\delta=1$ and $a=0$, (1.2) represents Weibull distribution [see Johnson and Kotz (1970, p. 250)].

(v) For $g(x)=x^2/2$, $\delta=1/2$ and $a=0$, (1.2) comes out to be half-normal distribution [see Davis (1952)].

(vi) For $g(x) = x^2$, $\delta = 1$ and $a = 0$, (1.2) is Rayleigh distribution [see Sinha (1986, p. 200)].

(vii) For $g(x) = x^2/2$, $\delta = \alpha/2$ and $a = 0$, (1.2) turns out to be Chi-distribution [see Patel, Kapadia and Owen (1976, p. 173)]. Taking $\alpha = 3$, it becomes Maxwell's distribution [see Tyagi and Bhattacharya (1989a, b)].

(viii) For $g(x) = \log(1 + x^b)$, $\delta = 1$ and $a = 0$, we obtain from (1.2) Burr distribution [see Burr (1942) and Cislak and Burr (1968)].

(ix) For $g(x) = (\log x)^2/2$, $\delta = 1/2$ and $a = 0$, (1.2) leads us to log-normal distribution [see Johnson and Kotz (1970, p.112).

(x) For $g(x) = \log x$, $\delta = 1$ and $a = 1$, (1.2) leads us to Pareto distribution [see Johnson and Kotz (1970, p. 233)].

It is to be noted here that the distributions (i) – (vii) follow as specific cases of the model (1.1) [see Chaturvedi, Kumar and Kumar (1998)], whereas, the distributions (viii) - (x) are additional models which are incorporated in (1.2).

In Sections 2 and 3, respectively, we consider SPRTS for testing simple hypotheses (versus simple alternatives) for the parameter θ (when δ is known) and for the parameter δ (when θ is known). The robustness of the proposed SPRTS is studied when the distribution under consideration has undergone a change. In Section 4, sequential test for the composite hypothesis regarding δ (when θ is unknown) is developed and the OC and ASN functions are obtained. In order to calculate the roots of the equation needed to evaluate OC and ASN functions, we give a method of choosing the initial values in order to apply Newton-Raphson method. Finally, in Section 5, a discussion on the results is provided.

2 Robustness of the SPRT for testing the hypothesis regarding θ when δ is known

Given a sequence of observations $X_1, X_2, \ldots$ from (1.2), suppose one wishes to test the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1$ $(> \theta_0)$, when 'δ' is known. The SPRT for testing H_0 is defined as follows:

Let

$$Z_i = \ln\left\{\frac{f(X_i;\delta,\theta_1)}{f(X_i;\delta,\theta_0)}\right\} = \delta \ln\left(\frac{\theta_0}{\theta_1}\right) + \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) g(X_i). \tag{2.1}$$

We choose two numbers A and B such that $0 < B < 1 < A$. At the nth stage, accept H_0 if $\sum_{i=1}^{n} Z_i \leq \ln B$, reject H_0 if $\sum_{i=1}^{n} Z_i \geq \ln A$, otherwise, continue sampling by taking the $(n+1)$th observation. If $\alpha \in (0,1)$ and $\beta \in (0,1)$ are type I and type II errors, respectively, then according to Wald (1947), A and B are approximately given by

$$A \approx (1-\beta)/\alpha \text{ and } B \approx \beta/(1-\alpha). \tag{2.2}$$

The OC function is approximately given by

$$L(\theta) \approx \frac{A^{t_0}-1}{A^{t_0}-B^{t_0}},$$

where t_0 is the nonzero solution of the equation

$$E(e^{t_0 Z_i}) = 1. \tag{2.3}$$

Using the fact that $g(X_i)$ follows a gamma distribution with scale parameter θ and shape parameter δ, we obtain from (2.1) and (2.3) that

$$\left[\frac{(\theta_0/\theta_1)^{t_0}}{\{1-\theta t_0(1/\theta_0 - 1/\theta_1)\}}\right]^{\delta} = 1,$$

or,

$$\theta = \frac{1-(\theta_0/\theta_1)^{t_0}}{t_0(1/\theta_0 - 1/\theta_1)}.$$

The ASN function is approximately given by

$$E(N|\theta) \approx \frac{L(\theta)\ln B + \{1-L(\theta)\}\ln A}{E(Z_i|\theta)},$$

which on using (2.1) gives that

$$E(N|\theta) \approx \frac{L(\theta)\ln B + \{1-L(\theta)\}\ln A}{\delta\{\ln(\theta_0/\theta_1) + \theta(1/\theta_0 - 1/\theta_1)\}}. \tag{2.4}$$

From (2.4), the ASN function under H_0 and H_1 is given [see Siegmund (1985, p.13, Remarks 2.20)], respectively, by

$$E_0(N) \approx \frac{(1-\alpha)\ln B + \alpha \ln A}{\delta\{\ln(\theta_0/\theta_1) + (1-\theta_0/\theta_1)\}}$$

and

$$E_1(N) \approx \frac{\beta \ln B + (1-\beta)\ln A}{\delta\{\ln(\theta_0/\theta_1) + (\theta_1/\theta_0 - 1)\}}.$$

The maximum value of the ASN occurs for $\theta = \theta^*$, where θ^* is the solution of $E(Z_i|\theta) = 0$ and this value is given by [see Rohatgi (1976, p.635)]

$$E_{\theta^*}(N) \approx (-\ln A \ln B)/E(Z_i^2|\theta^*). \tag{2.5}$$

It is easy to see that

$$\theta^* = \ln(\theta_1/\theta_0)/(1/\theta_0 - 1/\theta_1)$$

and

$$E(Z_i^2|\theta^*) = \delta\{\ln(\theta_0/\theta_1)\}^2. \tag{2.6}$$

Using (2.6) in (2.5), we obtain that

$$E_{\theta^*}(N) \approx (-\ln A \ln B)/\delta\{\ln(\theta_0/\theta_1)\}^2.$$

Now we study the robustness of the SPRT. Let us now suppose that the parameter δ has undergone a change and the pdf (1.2) becomes $f(x;\gamma,\theta)$, which is obtained on replacing δ by γ. The OC function of the SPRT is

$$L(\theta) \approx \frac{A^h - 1}{A^h - B^h}, \tag{2.7}$$

where, in order to study the robustness of the SPRT, we consider h as the solution of the equation

$$\int_a^\infty \left\{\frac{f(x_i;\delta,\theta_1)}{f(x_i;\delta,\theta_0)}\right\}^h f(x_i;\gamma,\theta)\mathrm{d}x_i = 1. \tag{2.8}$$

Using (1.2) and denoting by $\phi_1 = (\delta/\gamma)$, we obtain from (2.8) that

$$\frac{1}{\theta^\gamma\Gamma(\gamma)}\left(\frac{\theta_0}{\theta_1}\right)^{\delta h}\int_a^\infty \exp\left\{-\left(\frac{1}{\theta} - \frac{h}{\theta_0} + \frac{h}{\theta_1}\right)g(x_i)\right\} g^{\gamma-1}(x_i)g'(x_i)\mathrm{d}x_i = 1,$$

or,

$$\left\{\left(\frac{\theta_0}{\theta_1}\right)^{\phi_1 h} / \theta\left(\frac{1}{\theta} - \frac{h}{\theta_0} + \frac{h}{\theta_1}\right)\right\} = 1,$$

or,

$$\theta = \left\{1 - \left(\frac{\theta_0}{\theta_1}\right)^{\phi_1 h}\right\} / \left\{h\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\right\} \tag{2.9}$$

While dealing with the expression (2.9), we faced the practical problem of choosing the initial values in order to get the roots h through Newton-Raphson method. We give below a method to 'asses' the roots. To this end, we rewrite (2.9) as

$$\phi_1 h \ln\left(\frac{\theta_0}{\theta_1}\right) = \ln\left\{1 - \theta h\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\right\}.$$

Using the expansion for $\ln(1-x)$ and retaining the terms up to third degree in h and simplifying, we get

$$\frac{\theta^3 h^2}{3}\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)^3+\frac{\theta^2 h}{2}\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)^2+\theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)+\phi_1\ln\left(\frac{\theta_0}{\theta_1}\right)=0,$$

or,

$$h=\frac{-3}{4\theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)}\pm\frac{3}{2\theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)}\left\{-\frac{13}{12}-\frac{4\phi_1\ln(\theta_0/\theta_1)}{3\theta\left(\frac{1}{\theta_0}-\frac{1}{\theta_1}\right)}\right\}^{1/2}. \quad (2.10)$$

We use the values of h given by (2.10) as initial values for solving (2.9) through Newton-Raphson method. The satisfactory performance of the method is explained in Section 5.

3 Robustness of the SPRT for testing the hypothesis regarding δ when θ is known

Let $X_1, X_2, \ldots$ be a sequence of iid observations from (1.2). Our goal is to test the simple hypothesis $H_0 : \delta = \delta_0$ against the simple alternative $H_1 : \delta = \delta_1$ $(> \delta_0)$, when θ is known. We propose the following SPRT:

Define

$$\begin{aligned} Z_i &= \ln\left\{\frac{f(X_i;\delta_1,\theta)}{f(X_i;\delta_0,\theta)}\right\} \\ &= \ln\Gamma(\delta_0)-\ln\Gamma(\delta_1)+(\delta_1-\delta_0)\ln g(X_i)+(\delta_0-\delta_1)\ln\theta. \quad (3.1) \end{aligned}$$

Let us denote by $\nu(n,\delta_0,\delta_1) = n\{\ln\Gamma(\delta_0)-\ln\Gamma(\delta_1)\}+n(\delta_0-\delta_1)\ln\theta$. At the nth stage, accept H_0 if

$$\sum_{i=1}^{n}\ln g(X_i) \geq (\delta_1-\delta_0)^{-1}[\ln B-\nu(n,\delta_0,\delta_1)],$$

reject H_0 if

$$\sum_{i=1}^{n}\ln g(X_i) \geq (\delta_1-\delta_0)^{-1}[\ln A-\nu(n,\delta_0,\delta_1)],$$

and continue sampling by taking the $(n+1)$th observation if

$$(\delta_1-\delta_0)^{-1}[\ln B-\nu(n,\delta_0,\delta_1)] < \sum_{i=1}^{n}\ln g(X_i) < (\delta_1-\delta_0)^{-1}[\ln A-\nu(n,\delta_0,\delta_1)].$$

Here, A and B are same as that defined at (2.2). The OC function is defined by

$$L(\delta) \approx (A^{t_0} - 1)/(A^{t_0} - B^{t_0}).$$

where t_0 is the nonzero solution of

$$E(e^{t_0 Z_i}) = 1. \tag{3.2}$$

Once again, using the fact that $g(X_i)$ follows a gamma distribution with scale parameter θ and shape parameter 'δ', we obtain from (3.2) that

$$\{\Gamma(\delta_0)/\Gamma(\delta_1)\}^{t_0}\{\Gamma(\delta + t_0(\delta_1 - \delta_0))/\Gamma(\delta)\} = 1.$$

It is interesting to note that t_0, and hence, the OC function is free from θ. The ASN function is approximately given by

$$E(N|\delta) \approx [L(\delta)\ln B + \{1 - L(\delta)\}\ln A/E(Z_i|\delta)]. \tag{3.3}$$

Using a result of Gradshteyn and Ryzhik (1965, p. 576, §4.352 (4)) that

$$\int_0^\infty x^{\mu-1}e^{-x}\ln x \, \mathrm{d}x = \Gamma'(\mu), \tag{3.4}$$

where $\Gamma'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Gamma(x)$, we obtain from (3.1) that

$$E(Z_i|\delta) = \ln\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\} + (\delta_1 - \delta_0)\Gamma'(\delta). \tag{3.5}$$

From (3.3) and (3.5), the ASN function under H_0 and H_1 is given [see Siegmund (1985, p.13, Remarks 2.20)], respectively, by

$$E_0(N) \approx \frac{[(1-\alpha)\ln B + \alpha \ln A]}{[\ln\Gamma(\delta_0) - \ln\Gamma(\delta_1) + (\delta_1 - \delta_0)\Gamma'(\delta_0)]}$$

and

$$E_1(N) \approx \frac{[\beta \ln B + (1-\beta)\ln A]}{[\ln\Gamma(\delta_0) - \ln\Gamma(\delta_1) + (\delta_1 - \delta_0)\Gamma'(\delta_1)]}.$$

The maximum value of the ASN function is at the point $\delta = \delta^*$, where δ^* is the solution of $E(Z_i|\delta) = 0$, i.e.,

$$\Gamma'(\delta^*) = \{\ln\Gamma(\delta_1) - \ln\Gamma(\delta_0)\}/(\delta_1 - \delta_0)$$

and this maximum value is given by [see Rohatgi (1976, p.635)]

$$E_{\delta^*}(N) \approx (-\ln A \ln B)/E(Z_i^2|\delta^*).$$

Utilizing (3.1), (3.4) and a result of Gradshteyn and Ryzhik (1965, §4.358, p. 578), we get

$$E(Z_i^2|\delta^*) = 2\{\ln(\Gamma(\delta_0)/\Gamma(\delta_1))\}^2 + \frac{(\delta_1-\delta_0)^2}{\theta^{\delta^*}\Gamma(\delta^*)}\xi(2,\delta^*-1),$$

where $\xi(z,q)$ is Riemann's Zeta function defined by

$$\xi(z,q) = \sum_{n=0}^{\infty}\frac{1}{(q+n)^2}\ .$$

Let us now suppose that the distribution (1.2) has undergone a change and it becomes $f(x;\delta,\eta)$, which is obtained on replacing θ by η. The OC function of the SPRT is

$$L(\delta) \approx (A^h-1)/(A^h-B^h),$$

where 'h' is the solution of the equation

$$\int_a^{\infty}\left\{\frac{f(x_i;\delta_1,\theta)}{f(x_i;\delta_0,\theta)}\right\}^h f(x_i;\delta,\eta)\mathrm{d}x_i = 1. \tag{3.6}$$

Using (3.1), we obtain from (3.6) that

$$\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^h \frac{\theta^{h(\delta_0-\delta_1)}}{\eta^{\delta}\Gamma(\delta)}\int_0^{\infty} y^{\delta+h(\delta_1-\delta_0)-1}e^{-y/\eta}\mathrm{d}y = 1,$$

or,

$$\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^h \left\{\frac{\Gamma(\delta+h(\delta_1-\delta_0))}{\Gamma(\delta)}\right\}\phi_2^{h(\delta_1-\delta_0)} = 1, \tag{3.7}$$

where $\phi_2 = (\eta/\theta)$. The expression for the ASN function is same as that given at (3.5), where

$$E(Z_i|\delta) = \ln(\Gamma(\delta_0)/\Gamma(\delta_1)) + (\delta_1-\delta_0)\ln\phi_2 + (\delta_1-\delta_0)\Gamma'(\delta). \tag{3.8}$$

The expressions (3.7) and (3.8) are not of much use for the purpose of calculating the OC and ASN functions. For practical purpose, we use the approximation [see Phatarfod (1971, p.876 (2.3))]

$$\ln\Gamma(x) \approx \ln(\sqrt{2\pi}) - x + (x-1/2)\ln x, \tag{3.9}$$

i.e.

$$\Gamma'(x) \approx \ln x - 1/(2x). \tag{3.10}$$

Expression (3.7) now gives that

$$h[(\delta_0 - 1/2)\ln\delta_0 - (\delta_1 - 1/2)\ln\delta_1 + (\delta_1 - \delta_0)\ln\phi_2] - (\delta - 1/2) \\ \ln\delta + [\delta + h(\delta_1 - \delta_0) - 1/2]\ln[\delta + h(\delta_1 - \delta_0)] = 0. \tag{3.11}$$

Using the expansion for $\ln(1+x)$, retaining the terms up to third-degree in h and simplifying, we obtain from (3.11) that

$$\frac{(\delta+1)(\delta_1-\delta_0)^3}{6\delta^3}h^2 - \frac{(2\delta+1)(\delta_1-\delta_0)^2}{6\delta^2}h - \{(\delta_0 - 1/2)\ln\delta_0 \\ -(\delta_1 - 1/2)\ln\delta_1 + (\delta_1-\delta_0)\ln\delta + \left(1 + \ln\phi_2 - \frac{1}{2\delta}\right)(\delta_1 - \delta_0) = 0,$$

or,

$$\begin{aligned} h &= \frac{3\delta}{(\delta+1)(\delta_1-\delta_0)}\left[(2\delta+1) \pm \frac{\delta}{(\delta_1-\delta_0)}\left[\frac{(2\delta+1)^2(\delta_1-\delta_0)^2}{16\delta^2}\right.\right. \\ &\quad + \frac{4(\delta+1)(\delta_1-\delta_0)}{6\delta}\left\{(\delta_0 - 1/2)\ln\delta_0 - (\delta_1 - 1/2)\ln\delta_1\right. \\ &\quad \left.\left.\left. + (\delta_1-\delta_0)\left(1 + \ln\delta\phi_2 - \frac{1}{2\delta}\right)\right\}\right]^{1/2}\right]. \end{aligned} \tag{3.12}$$

4 SPRT for testing the hypothesis regarding 'δ' when θ is unknown

Based on a sequence $X_1, X_2, \ldots$ of iid observations from (1.2), our goal is to test the composite hypothesis $H_0 : \delta = \delta_0$ versus $H_1 : \delta = \delta_1$, where θ is assumed to be unknown. We proceed as follows:

Let us make the transformations

$$Y_r = g(X_{r+1}) / \sum_{i=1}^{r+1} g(X_i), \quad r = 1, 2, \ldots, (n-1). \tag{4.1}$$

Since $g(X_i)$'s are iid rv's, each having gamma distribution with scale parameter θ and shape parameter δ, utilizing the additive property of gamma distributions [see Johnson and Kotz (1970, p.181)], we conclude from (4.1) that Y_r follows a beta distribution of the first kind, with pdf given by

$$f(y_r; \delta) = (B(\delta, r\delta))^{-1} y_r^{\delta-1}(1-y_r)^{r\delta-1}; \; 0 < y_r < 1. \tag{4.2}$$

We first establish the independence of Y_r's. To this end, denoting by

$$U_r = g(X_{r+1}) / \sum_{i=1}^{r} g(X_i),$$

we can write

$$Y_r = U_r/(1+U_r),\ r = 1, 2, \ldots, (n-1). \tag{4.3}$$

From (4.3), the independence of Y_r's follow, if we can prove that U_r and U_{r-1} are independent. We note that the joint distribution of $V = g(X_r)$ and $W = \sum_{i=1}^{r-1} g(X_i)$ is

$$f(v,w) = \frac{v^{\delta-1} w^{(r-1)\delta-1} \exp(-(v+w)/\theta)}{\Gamma(\delta)\Gamma((r-1)\delta)\theta^{r\delta}}. \tag{4.4}$$

Making the transformations $S = V + W$ and $Z = V/W$, from (4.4), the joint distribution of S and Z comes out to be

$$g(s,z) = \left\{\frac{s^{r\delta-1}\exp(-s/\theta)}{\Gamma(r\delta)\theta^{r\delta}}\right\}\left\{\frac{z^{\delta-1}}{B(\delta,(r-1)\delta)(1+z)^{r\delta}}\right\}. \tag{4.5}$$

From (4.5), we conclude that the rv's $g(X_{r+1})$, $\sum_{i=1}^{r} g(X_i)$ and U_{r-1} are mutually independent. This establishes the independence of U_r and U_{r-1}. It is to be noted here that the rv's Y_r's are not identically distributed. From (4.2), the likelihood of observing $Y_1, \ldots, Y_{n-1}$ is

$$\begin{aligned} R_n &= \left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^{n-1} \prod_{r=1}^{n-1}\left\{\frac{\Gamma(r\delta_0)\Gamma((r+1)\delta_1)}{\Gamma(r\delta_1)\Gamma((r+1)\delta_0)}\right\} y_r^{(\delta_1-\delta_0)}(1-y_r)^{r(\delta_1-\delta_0)} \\ &= \left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^{n}\left\{\frac{\Gamma(n\delta_1)}{\Gamma(n\delta_0)}\right\}\prod_{r=1}^{n-1} y_r^{(\delta_1-\delta_0)}(1-y_r)^{r(\delta_1-\delta_0)}. \end{aligned} \tag{4.6}$$

The sequential test is carried out as follows:

Let us denote by $Z_n = \ln R_n$. For A and B defined at (2.2), at the nth stage, accept H_0 if $Z_n \le \ln B$, reject H_0 if $Z_n \ge \ln A$, otherwise, continue sampling by taking the $(n+1)$th observation. In terms of the original observations, the continuation region is given by

$$\begin{aligned} \ln B + n\ln\{\Gamma(\delta_1)/\Gamma(\delta_0)\} + f(n) &< (\delta_1-\delta_0)\sum_{r=1}^{n}\ln(g(X_r)/\overline{g}(n) \\ &< \ln A + n\ln\{\Gamma(\delta_1)/\Gamma(\delta_0)\} + f(n), \end{aligned} \tag{4.7}$$

where

$$f(n) = n(\delta_1-\delta_0)\ln(n) + \ln\{\Gamma(n\delta_0)/\Gamma(n\delta_1)\} \tag{4.8}$$

and

$$\overline{g}(n) = n^{-1}\sum_{i=1}^{n} g(X_i).$$

Using the approximation [see Phatarfod (1971, p.876 (2.3))]

$$\ln\Gamma(x) \approx \ln(\sqrt{2\pi}) - x + (x - 1/2)\ln x, \tag{4.9}$$

we obtain from (4.8) that

$$f(n) \approx n[(\delta_1 - \delta_0) + \delta_0 \ln\delta_0 - \delta_1 \ln\delta_1] + (1/2)\ln(\delta_1/\delta_0). \tag{4.10}$$

From (4.7), we note that the boundaries for $(\delta_1 - \delta_0)\sum_{r=1}^{n} \ln(g(X_r)/\overline{g}(n)$ are not straight lines. However, on making use of (4.10), the boundaries can be reduced to straight lines.

Utilizing (4.6), for a fixed positive integer n, the moment generating function of Z_n comes out to be

$$\begin{aligned} & M_n(t) = E\{\exp(tZ_n)\} \\ = & \left[\left\{\frac{\Gamma(n\delta_1)}{\Gamma(n\delta_0)}\right\}\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^n\right]^t \prod_{r=1}^{n-1}\left\{\frac{B(t(\delta_1-\delta_0)+\delta, rt(\delta_1-\delta_0)+r\delta)}{B(\delta, r\delta)}\right\} \\ = & \left[\left\{\frac{\Gamma(n\delta_1)}{\Gamma(n\delta_0)}\right\}\left\{\frac{\Gamma(\delta_0)}{\Gamma(\delta_1)}\right\}^n\right]^t \left\{\frac{\Gamma(n\delta)\{\Gamma(t(\delta_1-\delta_0)+\delta)\}^n}{\Gamma^n(\delta)\Gamma(nt(\delta_1-\delta_0)+n\delta)}\right\}. \end{aligned} \tag{4.11}$$

Denoting by N, the stopping rule associated with the sequential procedure, we know Wald's fundamental identity [see Wald (1947, p. 160)]

$$E[\exp(tZ_N + \ln(M_n(t))^{-1}] = 1, \tag{4.12}$$

(4.12) where $M_N(t)$ is obtained from (4.11) replacing n by N. Of course, the identity (4.12) is exact but is of no practical use. Once again, utilizing the approximation (4.9), we obtain after some algebraic manipulations that

$$\begin{aligned} \ln(M_N(t))^{-1} \approx & \ln\left[\frac{\delta^\delta}{\Gamma(\delta)}\left\{\frac{\Gamma(t(\delta_1-\delta_0)+\delta)}{(t(\delta_1-\delta_0)+\delta)^{(t(\delta_1-\delta_0)+\delta)}}\right\}\left(\frac{\delta_1^{\delta_1}\Gamma(\delta_0)}{\delta_0^{\delta_0}\Gamma(\delta_1)}\right)^t\right]^{-N} \\ & +\ln\left[\frac{\delta}{(t(\delta_1-\delta_0)+\delta)}\left(\frac{\delta_1}{\delta_0}\right)^t\right]^{1/2}. \end{aligned} \tag{4.13}$$

Making substitution from (4.13) in (4.12), we get

$$E[e^{tZ_N}\{\lambda(t)\}^{-N}] \approx \gamma(t), \tag{4.14}$$

where

$$\lambda(t) = \frac{\delta^\delta}{\Gamma(\delta)}\left\{\frac{\Gamma(t(\delta_1-\delta_0)+\delta)}{(t(\delta_1-\delta_0)+\delta)^{(t(\delta_1-\delta_0)+\delta)}}\right\}\left(\frac{\delta_1^{\delta_1}\Gamma(\delta_0)}{\delta_0^{\delta_0}\Gamma(\delta_1)}\right)^t \tag{4.15}$$

and

$$\gamma(t) = \left[\frac{(t(\delta_1 - \delta_0) + \delta)}{\delta} \left(\frac{\delta_0}{\delta_1}\right)^t\right]^{1/2}. \tag{4.16}$$

Let $t \equiv t(\delta)$ be the nonzero solution of the equation $\lambda(t) = 1$, i.e. from (4.15),

$$G(t(\delta_1 - \delta_0) + \delta) + t[G(\delta_0) - G(\delta_1)] = G(\delta), \tag{4.17}$$

where

$$G(x) = \ln \Gamma(x) - x \ln x.$$

Once again using (4.9), we obtain from (4.17) that

$$\delta = (t(\delta_1 - \delta_0))/\{(\delta_1/\delta_0)^t - 1\}. \tag{4.18}$$

For the values of $t(\delta)$ and δ satisfying (4.18), the OC function is given by

$$L(\delta) \approx (A^{t(\delta)} - \gamma(t(\delta)))/(A^{t(\delta)} - B^{t(\delta)}).$$

Now we obtain the expression for the ASN function. To this end, differentiating (4.14) with respect to t, we get

$$E\left[Z_N e^{tZ_N}\{\lambda(t)\}^{-N} - N\{\lambda(t)\}^{-(N+1)}\lambda'(t)e^{tZ_N}\right] \approx \gamma'(t),$$

which on substituting $t = 0$ gives that

$$E(N) \approx \{E(Z_N) - \gamma'(0)\}/\lambda'(0).$$

From (4.16),

$$\gamma'(0) = (1/2)[\ln(\delta_0/\delta_1) - (\delta_1 - \delta_0)/\delta']$$

and ignoring the excess over boundaries

$$E(Z_N) \approx L(\delta) \ln B + [1 - L(\delta)] \ln A.$$

Furthermore, using approximation (4.9) in (4.15) and taking limit as $t \to 0$, we get

$$\lambda'(0) \approx (1/2)[\ln(\delta_1/\delta_0) - (\delta_1 - \delta_0)/\delta].$$

5 Discussion

We first consider the problem of testing simple versus simple hypotheses regarding θ, when δ is known. Let us consider $H_0 : \theta = 25$ versus $H_1 : \theta = 30$ taking $\alpha = \beta = .05$. For different values of θ, the real roots h are obtained from (2.10) and are taken as initial values to solve the original equation (2.9) through Newton-Raphson method. The approximate values of h and the values obtained through Newton-Raphson method are presented in Table 1. It is interesting to note that the two values are quite close to each other, justifying the use of the approximation (2.10). The validity of the approximation is also verified by considering the values of the 'h' under H_0 and H_1 when $\phi_1 = 1$. It is noted that the values of h under H_0 and H_1 (marked with asterisks) are quite close to the true values 1 and -1, respectively. In order to study the robustness of the SPRT, we have plotted OC and ASN curves in Figure 1 and Figure 2, respectively, for $\phi_1 = 0.98$, 1 and 1.012. It is evident from the figures that for $\phi_1 > 1$ (< 1), the OC curve shifts to the right (left) of the curve corresponding to $\phi_1 = 1$, whereas, the ASN function curve shifts below (above) to the curve corresponding to $\phi_1 = 1$. Both the curves are highly sensitive for changes in δ. Thus we conclude that for the present model, the SPRT for testing the hypothesis regarding θ, is highly non-robust for changes in δ.

Table 1

Values of h

θ	$\phi_1 = 0.98$		$\phi_1 = 1.0$		$\phi_1 = 1.012$	
	Approximate	N-R Method	Approximate	N-R Method	Approximate	N-R Method
24.0	1.2833	1.2593	1.5029	1.4652	1.6310	1.5835
24.5	1.0338	1.0204	1.2527	1.2294	1.3801	1.3494
25.0	0.7945	0.7880	1.0131	1.0000*	1.1402	1.1218
25.5	0.5643	0.5619	0.7832	0.7767	0.9102	0.9002
26.0	0.3421	0.3416	0.5619	0.5593	0.6891	0.6845
26.5	0.1268	0.1268	0.3481	0.3475	0.4759	0.4743
27.0	-0.0826	-0.0826	0.1409	0.1409	0.2696	0.2694
27.5	-0.2873	-0.2869	-0.0607	-0.0607	0.0694	0.0694
28.0	-0.4884	-0.4863	-0.2577	-0.2574	-0.1258	-0.1270
28.5	-0.6872	-0.6810	-0.4512	-0.4495	-0.3169	-0.3162
29.0	-0.8854	-0.8713	-0.6424	-0.6371	-0.5050	-0.5025
29.5	-1.0845	-1.0573	-0.8327	-0.8206	-0.6913	-0.6845
30.0	-1.2871	-1.2392	-1.0236	-1.0000*	-0.8771	-0.8624
30.5	-1.4963	-1.4172	-1.2173	-1.1755	-1.0641	-1.0365
31.0	-1.7168	-1.5914	-1.4164	-1.3473	-1.2543	-1.2069

Now we consider the problem of testing the hypothesis $H_0 : \delta = 45$ versus $H_1 : \delta = 50$, when θ is known, taking $\alpha = \beta = .05$. For different values of δ and ϕ_2, the real roots h obtained from (3.12) and are taken as initial values in (3.7) and ASN and OC functions are evaluated with the help of (2.4) and (2.7), respectively. As exhibited by Table 2, the approximate solution (3.12) gives quite satisfactorily results as, for $\phi_2 = 1$, the values of h under H_0 and H_1 (marked with asterisks) are quite close

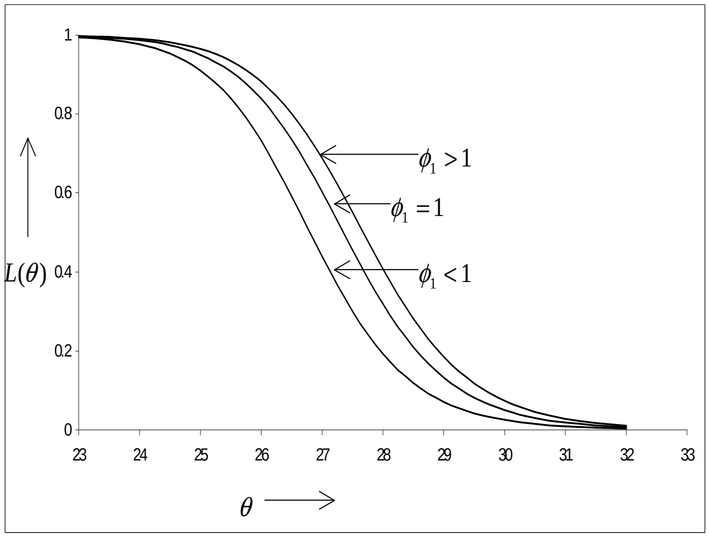

Figure 1
OC function curves for testing the hypothesis regarding θ (δ known)

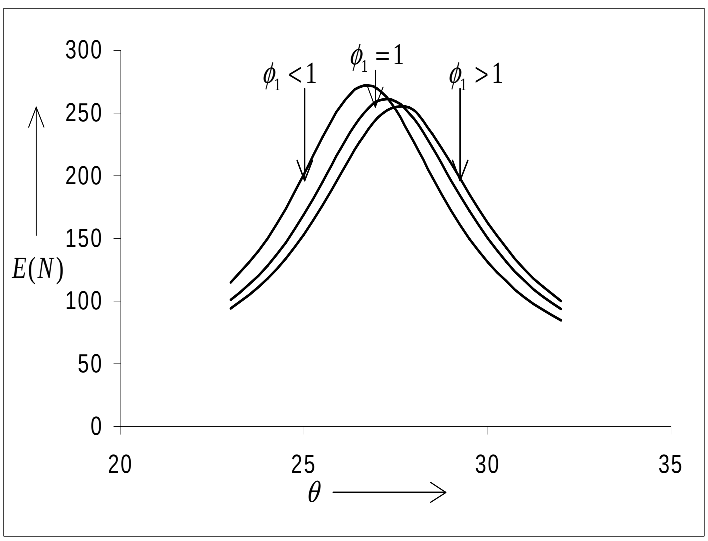

Figure 2
ASN function curves for testing the hypothesis regarding θ (δ known)

to exact values $+1$ and -1, respectively. In order to study the robustness of the SPRT, we have plotted OC and ASN curves in Figure 3 and Figure 4, respectively, for $\phi_2 = .99$, 1.00 and 1.025. The OC curve shifts to left (right) for $\phi_2 > 1$ (< 1) of the curve corresponding to $\phi_2 = 1$, whereas, the ASN curve shifts above (below) to the curve corresponding to $\phi_2 = 1$. Both the curves are highly sensitive for changes in θ.

Table 2

Values of h

δ	$\phi_2 = 0.99$	$\phi_2 = 1.0$	$\phi_2 = 1.025$
44.5	1.3610	1.2054	1.0340
45.0	1.1557	1.0008*	0.8301
45.5	0.9513	0.7972	0.6272
46.0	0.7479	0.5944	0.4252
46.5	0.5454	0.3925	0.2240
47.0	0.3438	0.1915	0.0235
47.5	0.1429	-0.0089	-0.1762
48.0	-0.0573	-0.2085	-0.3753
48.5	-0.2567	-0.4075	-0.5737
49.0	-0.4556	-0.6058	-0.7715
49.5	-0.6538	-0.8036	-0.9688
50.0	-0.8514	-1.0008*	-1.1656
50.5	-1.0485	-1.1975	-2.1906

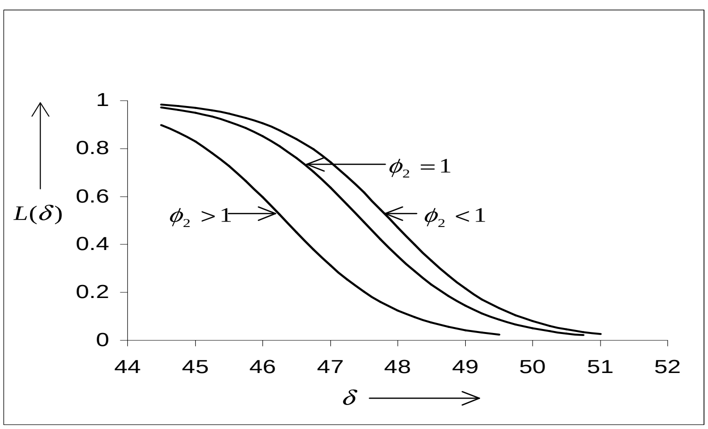

Figure 3
OC function curves for testing the hypothesis regarding δ (θ known)

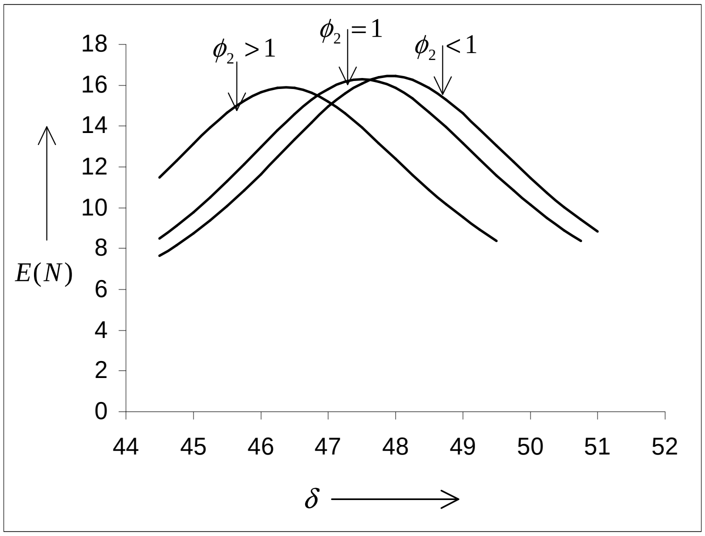

Figure 4
ASN function curves for testing the hypothesis regarding δ (θ known)

Finally, we consider the problem of testing the hypotheses for δ, assuming θ to be unknown. We consider $H_0 : \delta = 5$ versus $H_1 : \delta = 7$ taking $\alpha = \beta = .05$. The OC and ASN curves are plotted in Figure 5 and Figure 6, respectively.

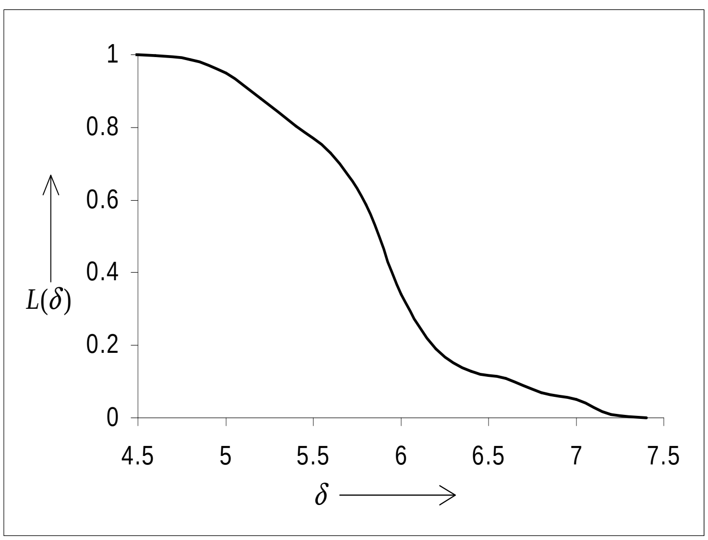

Figure 5
OC function curves for testing the hypothesis regarding δ (θ unknown)

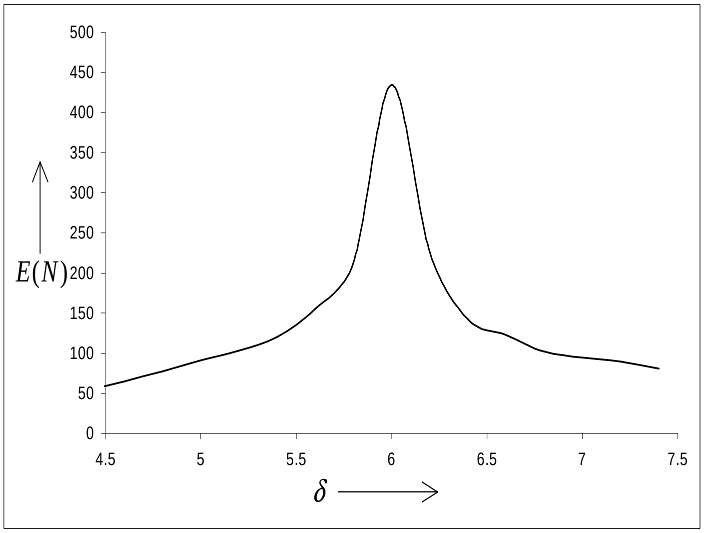

Figure 6
ASN function curves for testing the hypothesis regarding δ (θ unknown)

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References

Burr, I. W. (1942). Cumulative frequency functions. *Ann. Math. Statist.*, **13**, 215–232.

Chaturvedi, A., Kumar, A. and Kumar, S. (1998). Robustness of the sequential procedures for a family of life-testing models. *Metron*, **56**, 117–137.

Cislak, P. J. and Burr, I. W. (1968). On a general system of distributions I, its curve shape characteristic II, the sample median. *Jour. Amer. Statist. Assoc.*, **63**, 627.

Davis, D. J. (1952). The analysis of some failure data. *Jour. Amer. Statist. Assoc.*, **47**, 113–150.

Epstein, B. and Sobel, M. (1955). Sequential life tests in the exponential case. *Ann. Math. Statist.*, **26**, 82–93.

Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Table of Integrals, Series and Products*. London: Academic Press.

Harter, L. and Moore, A. H. (1976). An evaluation of exponential and Weibull test plans. *IEEE Trans. Reliab.*, **25**, 100–104.

Johnson, N. L. and Kotz, S. (1970). *Continuous Univariate Distributions I*. New York: John Wiley and Sons.

Montagne, E. R. and Singpurwalla, N. D. (1985). Robustness of the sequential exponential life-testing procedures. *Jour. Amer. Statist. Assoc.*, **80**, 715–719.

Phatarfod, R. M. (1971). A sequential test for gamma distribution. *Jour. Amer. Statist. Assoc.*, **66**, 876–878.

Patel. J. K., Kapadia, C. H. and Owen, D. B. (1976). *Handbook of Statistical Distributions*. New York: Marcel Dekker, Inc.

Rohatgi, V. K. (1976). *An Introduction to Probability Theory and Mathematical Statistics*. New York: John Wiley and Sons.

Siegmund, D. (1985). *Tests and Confidence Intervals*. New York: Springer -Verlag.

Sinha, S. K. (1986). *Reliability and Life-Testing*. New Delhi: Wiley Eastern Ltd.

Tyagi, R.K. and Bhattacharya, S.K. (1989a). A note on the MVU estimation of reliability for the Maxwell failure distribution. *Estadistica*, **41**, 73–79.

Tyagi, R.K. and Bhattacharya, S.K. (1989b). Bayes estimator of the Maxwell's velocity distribution function. *Statistica*, **XLIX**, 563–567.

Wald, A. (1947). *Sequential Analysis*. New York: John Wiley and Sons.