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Optimal conditional confidence interval for the shape parameter of a Weibull distribution

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Abstract: An optimal two-sided conditional confidence interval for the shape parameter β of a Weibull probability distribution is constructed. The construction follows the rejection of a preliminary test of significance for the null hypothesis: $\beta = \beta_0$ where β_0 is a fixed value. The bounds are derived according to the method set forth by Meeks and D'Agostino (1983) and subsequently used by Arabatzis *et al.*(1989) in a Gaussian model and more recently by Chiou and Han (1994,1995) for conditional estimation in exponential models. The derived conditional confidence interval also suits non large sample sizes since it is based on the small sample technique advocated in Wardell (1997) combined with the pivotal quantities proposed in Bain and Engelhardt (1981). The average width and the coverage probability of this conditional interval are compared with whose of the usual unconditional interval and the likelihood based interval through simulations. Simulation studies have shown that on average the conditional confidence interval maintains a good confidence level and have a smaller width than the corresponding unconditional interval and the likelihood based interval over a significant part of the parameter space. As in the finding of Chiou and Han, the results on the conditional interval width agree with whose of Arabatzis *et al.* and contrast with whose of Meeks and D'Agostino stating that the unconditional interval is always shorter than the conditional interval.

Key words: Average width, coverage probability, optimal interval estimator, rejection of a preliminary hypothesis, simulation, Weibull distribution.

1 Introduction

Preliminary test of significance estimators initiated in Bancroft (1944) are widely used in practice to improve efficiency of estimators. The estimation following preliminary test of significance is termed "inference procedure based on conditional specification". A closely related procedure is the estimation by interval following rejection of a preliminary hypothesis, see, for instance, Chiou and Han (1994, 1995), Meeks and D'Agostino (1983) and Arabatzis et al. (1995). The distinction between these two procedures has been pointed out in Mahdi (2000). For a detailed account on the use of preliminary test procedures, see, for instance, Jugde and Bock (1978), Mahdi (1993, 1984), Giles et al. (1992), and more recently, Rai and Srivastava (1998). It is worth nothing that preliminary test procedures require high computations which lead to analytically unmanageable formulas. Therefore the use of computer programming and numerical solutions are unavoidable to study the properties of pre-test estimators, as pointed out in Mahdi (1999). The effect of the use of conditional interval estimation, following rejection of a preliminary test, for the exponential scale and location parameters has been investigated in Chiou and Han (1994, 1995). The inference is based on a type II censored sample. For instance, in Chiou and Han (1994), a conditional confidence interval for the scale parameter θ of a two-parameter exponential distribution is constructed following a rejection of a pre-test, with significance level α, for H_0: $\theta = \theta_0$ versus $H_a : \theta \neq \theta_0$. The authors have shown that the conditional interval performs well when $\hat{\theta}/\theta_0$ is close to one and when α is not small. $\hat{\theta}$ is the minimum variance unbiased estimator of θ. Note that the conditional interval estimation based on two sample data from exponential populations has been recently investigated in Mahdi (2000).

In this paper, we take a further step by considering the problem of optimal interval estimation for the shape parameter β of a Weibull distribution when it is suspected that $\beta = \beta_0$ for some given β_0. The value β_0 depends on the study at hand and could be given by a previous analysis. In lifetime analysis, for instance, β_0 could the shape parameter for the lifetime distribution of manufactured items before the occurrence of a suspected change in the manufacturing process. The inference is based on a full sample and the pivot proposed in Bain and Engelhardt (1981, 1991). This optimal interval is compared in terms of width and coverage probability to a corresponding unconditional interval with same targeted confidence level. Furthermore, we compare these confidence intervals to the likelihood based confidence interval, recommended in Meeker and Escobar (1998). It is worth noting that the Weibull distribution is widely used in various fields and, particularly in reliability analyzes, lifetime modelling, fatigue of materials and hydrology. In Lifetime data analysis, the practical value of the Weibull distribution stems from its ability to describe failure distributions with many different commonly occurring shapes as pointed out in Meeker and Escobar (1998). This model is suitable for the analysis of data based on either complete or censored samples as it often occurs with lifetime and reliability data. In hydrology, for instance, the Weibull model has been found suitable for low stream-flow analysis. It gained in importance for hydrological analysis from its link to the Generalized Extreme Value (GEV) distribution which counts among the most used distributions in the field. Indeed, the Weibull distribution is a reverse (GEV) distribution, see, for instance, Hosking (1986). We organize this paper as follows. In Section 2 we state the considered problem and in Section 3 we derive the bounds for the unconditional interval for β. Results on width optimality are also given. Conditional confidence interval and minimum width conditional confidence interval are treated in Section 4 and 5, respectively. The expected width of the unconditional confidence interval for β^2 is given in Section 5. In Section 6, we consider the minimum width conditional confidence interval. Numerical methods for approximating the bounds of the optimal width conditional confidence interval for β are presented in Section 7. In Section 8, we give formula for the actual coverage probability of the unconditional interval. The likelihood based confidence interval for β is derived in Section 9. Simulations results are discussed in Section 10 and figures illustrating these results are presented in Appendix. Finally, we conclude in Section 11.

2 Statement of the problem

Suppose that $X_1, \cdots, X_n$ constitute a random sample from a Weibull distribution with shape parameter β. It is suspected that $\beta = \beta_0$ where β_0 is a prefixed value. The null hypothesis $H_0 : \beta = \beta_0$ versus the alternative $H_1 : \beta \neq \beta_0$ is then tested and in the case of rejection a two sided confidence interval for β is thus constructed. To test H_0 and to construct the confidence interval bounds we use the pivotal quantity derived in Bain and Engelhardt (1981), that is,

$$\frac{cn\beta^2}{\hat{\beta}^2} \sim \chi^2_{c(n-1)} \tag{2.1}$$

where $c = 0.822$ and $\hat{\beta}$ is the maximum likelihood estimator of β. Note that the pivot (2.1) depends only on the shape parameter and its estimate. Note also that it is stated in Bain and Engelhardt (1991) that for heavy censoring of order r, approximately, $\frac{2r\beta}{\hat{\beta}} \sim \chi^2_{2(r-1)}$, while for complete samples, relating $\frac{\beta^2}{\hat{\beta}^2}$ to a chi-squared distribution with $n-1$ degrees of freedom seems more appropriate. This relation is provided by equation (3) in Bain and Engelhardt (1981) which is obtained for $c = 1$ in equation (2.1). Both cases $c = 0.822$ and $c = 1$ are therefore considered in our study.

The critical region associated with the preliminary test of H_0 performed at the significance level α is given by

$$\mathcal{R} = \left\{ \hat{\beta} : \ \frac{cn\beta_0^2}{\hat{\beta}^2} > \chi^2_{c(n-1)}(\alpha/2) \text{ or } \frac{cn\beta_0^2}{\hat{\beta}^2} < \chi^2_{c(n-1)}(1-\alpha/2) \right\}$$

where $\chi^2_\nu(\xi)$ denote the quantile of order $(1-\xi)100\%$ of the chi-square distribution with ν degrees of freedom. We derive below the bounds of the optimal width unconditional confidence interval for β.

3 Unconditional confidence interval for β

The usual upper bound β_U and lower bound β_L of the $100(1-p)\%$ unconditional confidence interval for β are given by

$$\beta_L = \hat{\beta}\sqrt{\frac{\chi^2_{c(n-1)}(1-p_1)}{cn}} \tag{3.1}$$

and

$$\beta_U = \hat{\beta}\sqrt{\frac{\chi^2_{c(n-1)}(p_2)}{cn}} \tag{3.2}$$

where $p = p_1 + p_2$. These bounds derive from the bounds of the unconditional confidence interval for β^2.

3.1 Optimal unconditional confidence interval

From the large sample property of the pivotal statistic (2.1), we state

Theorem 1 *For any p_1 and p_2 such that $0 < p = p_1 + p_2 < 1$, the minimum width $(1-p)100\%$ unconditional confidence interval bounds are obtained when $p_1 = p_2 = \dfrac{p}{2}$ in the case of a large sample size n.*

Proof. In the case of large samples, the probability density function of the pivotal quantity (2.1) is symmetrical around the vertical axis centered at $\nu = c(n-1)$. Consider now the intervals

$$I = \sqrt{\frac{\hat{\beta}^2\chi^2_{c(n-1)}(1-p/2)}{cn}} < \beta < \sqrt{\frac{\hat{\beta}^2\chi^2_{c(n-1)}(p/2)}{cn}} \tag{3.3}$$

and

$$I^{'} = \sqrt{\frac{\hat{\beta}^2\chi^2_{c(n-1)}(1-((k-1)p)/k)}{cn}} < \beta < \sqrt{\frac{\hat{\beta}^2\chi^2_{c(n-1)}(p/k)}{cn}} \tag{3.4}$$

where $k > 2$. Both I and I' are $(1-p)100\%$ confidence intervals for β. Let LI and LI' denote the width of I and I', respectively. To prove that $LI \leq LI'$, it suffices to prove that

$$\chi^2_{c(n-1)}(p/2) - \chi^2_{c(n-1)}(1-p/2) \leq \chi^2_{c(n-1)}(p/k) - \chi^2_{c(n-1)}(1-((k-1)p)/k) \tag{3.5}$$

which is equivalent to prove inequality,

$$\chi^2_{c(n-1)}(1-((k-1)p)/k)) - \chi^2_{c(n-1)}(1-p/2)) \leq \chi^2_{c(n-1)}(p/k) - \chi^2_{c(n-1)}(p/2). \tag{3.6}$$

But the intervals $[\chi^2_{c(n-1)}(1-p/2)), \chi^2_{c(n-1)}(1-((k-1)p)/k))]$ and $[\chi^2_{c(n-1)}(p/2), \chi^2_{c(n-1)}(p/k)]$ intercept the same area $\dfrac{(k-2)p}{2k}$ under the probability density function of $\chi^2_{c(n-1)}$ and that the curve of the probability density function is higher above the interval $[\chi^2_{c(n-1)}(1-p/2)), \chi^2_{c(n-1)}(1-((k-1)p)/k))]$. This proves inequality (3.6) and therefore Theorem **1** in the case $k > 2$. The proof in the case $1 < k < 2$ can be done in a similar way.

Corollary 1 *The two-sided minimum width unconditional $(1-p)100\%$ confidence interval for β has a smaller width then the one-sided unconditional interval of the form $(0, \hat{\beta}\sqrt{\dfrac{\chi^2_{c(n-1)}(p)}{cn}}]$ for large sample sizes n.*

Proof. The intervals $(0, \chi^2_{c(n-1)}(1-p/2))]$ and $[\chi^2_{c(n-1)}(p), \chi^2_{c(n-1)}(p/2))]$ have the same probability value but the width of the former is larger according to the position of the points 0, $\chi^2_{c(n-1)}(1-p/2)), \chi^2_{c(n-1)}(p))$ and $\chi^2_{c(n-1)}(p/2))$ under the curve of the probability density function of the $\chi^2_{c(n-1)}$ variable.

Remark 1 *In the case of small to moderate sample sizes, the distribution of the pivotal statistic* (2.1) *is asymmetric and often skewed to the right.*

Nevertheless, we have

Remark 2 *For small to moderate sample sizes, the one sided unconditional* $(1-p)100\%$ *confidence interval has a smaller width than the corresponding two-sided interval for significance levels p satisfying*

$$\chi^2_{c(n-1)}(1-((k-1)p)/k)) - \chi^2_{c(n-1)}(p/k) + \chi^2_{c(n-1)}(p) < 0. \qquad (3.7)$$

4 Conditional confidence interval for β

The bounds β_L^C and β_U^C of the $(1-p)100\%$ conditional confidence interval for β are computed using the conditional sampling distribution of the statistic $Z = \hat{\beta}^2$ given rejection of H_0. The conditional cumulative function of Z, given rejection of H_0, is $F_C(z) = P[Z \leq z | Z < z_1 \text{ or } Z > z_2]$ where $z_1 = \dfrac{cn\beta_0^2}{\chi^2_{c(n-1)}(\alpha/2)}$ and $z_2 = \dfrac{cn\beta_0^2}{\chi^2_{c(n-1)}(1-\alpha/2)}$. Thus,

$$F_C(z) = \begin{cases} \dfrac{F(z)}{1+F(z_1)-F(z_2)} & if z < z_1 \\ \\ \dfrac{F(z_1)}{1+F(z_1)-F(z_2)} & if z_1 \leq z < z_2 \\ \\ \dfrac{F(z)+F(z_1)-F(z_2)}{1+F(z_1)-F(z_2)} & if z \geq z_2 \end{cases} \qquad (4.1)$$

The corresponding probability density function is

$$f_C(z) = \begin{cases} \dfrac{f(z)}{1+F(z_1)-F(z_2)} & if z < z_1 \text{ or } z > z_2 \\ \\ 0 & \text{elsewhere} \end{cases} \qquad (4.2)$$

where

$$f(z) = (\beta\sqrt{(\frac{cn}{2})})^{c(n-1)}[\Gamma(\frac{c(n-1)}{2})]^{-1} z^{-(1+\frac{c(n-1)}{2})} \exp -[\frac{cn\beta^2}{2z}] \qquad (4.3)$$

and F is the cumulative function of the variable $Z = \dfrac{cn\beta^2}{Y}$ such that Y is a chi-square variable with $c(n-1)$ degrees of freedom. Note that,

$$F(z) = 1 - G(\frac{cn\beta^2}{z}), \qquad (4.4)$$

where G is the cumulative function of the chi-square variable with $c(n-1)$ degrees of freedom. Note also that F and G are respectively decreasing and increasing functions of the parameters β^2 and β. Therefore the values β_L^C and β_U^C are solutions of the system

$$F_C(\hat{z};\beta_U^C) = p_1 \tag{4.5}$$

and

$$F_C(\hat{z};\beta_L^C) = 1 - p_2 \tag{4.6}$$

or

$$G_C(\hat{z};\beta_U^C) = G_C(\frac{cn{\beta_U^C}^2}{\hat{z}}) = 1 - p_1 \tag{4.7}$$

and

$$G_C(\hat{z};\beta_L^C) = G_C(\frac{cn{\beta_L^C}^2}{\hat{z}}) = p_2 \tag{4.8}$$

where G_C is the conditional distribution of the variable Y given that, $Y < y_1 = \Psi\chi^2_{c(n-1)}(1-\alpha/2)$ or $Y > y_2 = \Psi\chi^2_{c(n-1)}(\alpha/2)$, that is,

$$G_C(y) = \begin{cases} \dfrac{G(y)}{1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2))} & if y < y_1 \\ \\ \dfrac{G(y_1)}{1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2))} & if y_1 \leq y < y_2 \\ \\ \dfrac{G(y) + G(y_1) - G(y_2)}{1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2))} & if y \geq y_2 \end{cases} \tag{4.9}$$

with $\Psi = (\dfrac{\beta}{\beta_0})^2$. In the section below, we give the expected width of the unconditional $(1 - p_1 - p_2)100\%$ level confidence interval for β^2.

5 Expected width of the unconditional interval for β^2

To compute the average width $\overline{LUI}$ of the unconditional interval for β^2, we integrate the function $L(z) = \dfrac{\chi^2_{c(n-1)}(p_2) - \chi^2_{c(n-1)}(1-p_1)}{cn} z$ with respect to the probability density function given in formula (4.2). After integration and simplification, we get for $\alpha > 0$,

$$\overline{LUI} \quad = \quad \frac{\beta^2}{2\Gamma(\dfrac{c(n-1)}{2})} \frac{(\chi^2_{c(n-1)}(p_2) - \chi^2_{c(n-1)}(1-p_1))}{[1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2))]}$$

$$\cdot \left[\Gamma(\frac{c(n-1)-2}{2}) - I_\Gamma((\frac{c(n-1)-2}{2}), \frac{1}{2}\Psi\chi^2_{c(n-1)}(\alpha/2)) \right.$$
$$\left. + I_\Gamma((\frac{c(n-1)-2}{2}), \frac{1}{2}\Psi\chi^2_{c(n-1)}(1-\alpha/2)) \right] \tag{5.1}$$

where I_Γ is the incomplete gamma function defined in Pearson (1922, 1968) for instance. In the case $\alpha = 0$, $\overline{LUI} = 0$ since $f_C(z) = 0$ for all $z > 0$.

Partial check. First if we let $\alpha \to 1$, then the conditional distribution of the statistic $\hat{\beta}$ tends to the unconditional one, as it should be, since we tend to always reject H_0. Furthermore, the quantity $\eta = 1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2)) \to 1$. Thus, $\overline{LUI} \to \beta^2 \dfrac{\chi^2_{c(n-1)}(p_2) - \chi^2_{c(n-1)}(1-p_1)}{c(n-1)-2}$. On the other hand, from the asymptotic unbiasedness property of the maximum likelihood estimators, the expected value of $L(Z)$ based on the unconditional distribution of Z has the limiting value

$$E(L(Z)) = \frac{\chi^2_{c(n-1)}(p_2) - \chi^2_{c(n-1)}(1-p_1)}{cn}\beta^2. \tag{5.2}$$

Therefore, the ratio

$$\frac{\overline{LUI}}{E(L(Z))} = \frac{cn}{c(n-1)-2} \simeq 1 \tag{5.3}$$

for large n. Similarly, if we let $\alpha \to 0$, that is, we tend to never reject H_0, then both quantities

$$\eta = 1 + G(\Psi\chi^2_{c(n-1)}(1-\alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2)) \tag{5.4}$$

and

$$\Gamma(\frac{c(n-1)-2}{2}) - I_\Gamma((\frac{c(n-1)-2}{2}), \frac{1}{2}\Psi\chi^2_{c(n-1)}(\alpha/2))$$
$$+ I_\Gamma((\frac{c(n-1)-2}{2}), \frac{1}{2}\Psi\chi^2_{c(n-1)}(1-\alpha/2)) \tag{5.5}$$

tend towards 0. Therefore, $\overline{LUI}$ is undetermined as it is expected since the integrand function f_C has an undetermined form in such a case. The limiting value depends on the rate of convergence towards 0 of the two quantities (5.4, 5.5). We derive below the optimal width conditional confidence interval for β.

6 Minimum width conditional confidence interval

To obtain the bounds of the minimum width confidence interval at $(1-p)100\%$ nominal level, we minimize the two variable function

$$\delta(\beta^C_U, \beta^C_L) = \beta^C_U - \beta^C_L \tag{6.1}$$

subject to the constraints

$$g_1(\beta_U^C, \beta_L^C) = 1 - G_C(\frac{cn{\beta_U^C}^2}{\hat{z}}) + G_C(\frac{cn{\beta_L^C}^2}{\hat{z}}) - p \leq 0 \tag{6.2}$$

and

$$g_2(\beta_U^C, \beta_L^C) = -\beta_L^C \leq 0 \tag{6.3}$$

The sufficient Kuhn-Tucker conditions for optimality, see for instance, Taha (1992) or Wardell (1997), are given by

$$(1, -1) - (\lambda_1, \lambda_2) \begin{bmatrix} \frac{\partial p_1}{\partial \beta_U^C} & \frac{\partial p_2}{\partial \beta_L^C} \\ 0 & -1 \end{bmatrix} = 0, \tag{6.4}$$

$$\lambda_1(p_1 + p_2 - p) = 0 \tag{6.5}$$

and

$$\lambda_2 \beta_L^C = 0. \tag{6.6}$$

where λ_1 and λ_2 are Lagrange multipliers associated with constraints (6.2) and (6.3), respectively. Now two cases arise: the case where β_L^C is assumed to be on the boundary of the solution space and the case where it is strictly inside the solution space. We treat these cases separately.

Case 1. $\beta_L^C = 0$**:** in this case, $p_2 = 0$. The solution β_U^C is solution of the equation

$$G_C(\frac{cn{\beta_U^C}^2}{\hat{z}}) = 1 - p. \tag{6.7}$$

Case 2. $\beta_L^C > 0$**:** in this case $\lambda_2 = 0$ and the system of equations $(6.4 - 6.6)$ reduces to the system

$$\Psi_1(\beta_U^C, \beta_L^C) = \frac{\partial p_1}{\partial \beta_U^C} + \frac{\partial p_2}{\partial \beta_L^C} = 0 \tag{6.8}$$

and

$$\Psi_2(\beta_U^C, \beta_L^C) = p_1 + p_2 - p = 0 \tag{6.9}$$

The above system can be solved only numerically. We propose to use one of the two numerical methods displayed below.

7 Numerical methods

We first present the Newton's method which is outlined, for instance, in Taha (1992) or Burden and Faires (2001).

7.1 Newton's method

The approximative solution (β_U^C, β_L^C) for the system (6.8, 6.9) can be obtain by the Newton's method as solution of the iterative system

$$\begin{bmatrix} \beta_U^{C(k)} \\ \beta_L^{C(k)} \end{bmatrix} = \begin{bmatrix} \beta_U^{C(k-1)} \\ \beta_L^{C(k-1)} \end{bmatrix} - J^{-1} \begin{bmatrix} \Psi_1(\beta_U^{C(k-1)}, \beta_L^{C(k-1)}) \\ \Psi_2(\beta_U^{C(k-1)}, \beta_L^{C(k-1)}) \end{bmatrix}$$

where J^{-1} is the inverse of the 2×2 matrix

$$J = \begin{bmatrix} \dfrac{\partial \Psi_1}{\partial \beta_U^C} & \dfrac{\partial \Psi_1}{\partial \beta_L^C} \\ \dfrac{\partial \Psi_2}{\partial \beta_U^C} & \dfrac{\partial \Psi_2}{\partial \beta_L^C} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial p_1}{\partial \beta_U^C} & \dfrac{\partial p_2}{\partial \beta_L^C} \\ \dfrac{\partial^2 p_1}{(\partial \beta_U^C)^2} & \dfrac{\partial^2 p_2}{(\partial \beta_L^C)^2} \end{bmatrix} \tag{7.1}$$

evaluated at the point $(\beta_U^C, \beta_L^C) = (\beta_U^{C(k-1)}, \beta_L^{C(k-1)})$ obtained at iteration $(k-1)$ from a suitable starting point $(\beta_U^{C(0)}, \beta_L^{C(0)})$.

7.2 Heuristic method

To avoid the difficulty of finding a suitable starting point for the Newton's method, we propose the following. First, subdivide the interval $[0, p]$ into a large number, say M, of subintervals and consider all pairs $(p_1, p_2) = (p_i, p - p_i)$ for $i = 1, \cdots, M$. Then solve the system (4.7, 4.8) for each pair (p_1, p_2). Now, since each pair (p_1, p_2) provides a $(1-p)100\%$ confidence interval for β, we therefore choose the one with minimum width.

8 Coverage probability

The coverage probability of the conditional interval is $1 - p$ since it is derived under this confidence level. However, the actual coverage probability of the unconditional confidence interval has to be computed under the conditional probability function of the pivotal random variable (2.1), given rejection of H_0. This coverage probability is thus given by

$$CP = \int_{\chi^2_{c(n-1)}(1-p_1)}^{\chi^2_{c(n-1)}(p_2)} dG_C(y). \tag{8.1}$$

Note that this coverage probability may exceed the confidence level $1 - p$. For instance, in the case $p_1 = 0$, $p_2 = p$ and $\Psi\chi^2_{c(n-1)}(1 - \alpha/2) > \chi^2_{c(n-1)}(p)$, it is easy to see that the coverage probability is equal to

$$\frac{1 - p}{1 + G(\Psi\chi^2_{c(n-1)}(1 - \alpha/2)) - G(\Psi\chi^2_{c(n-1)}(\alpha/2))} \geq 1 - p. \tag{8.2}$$

From simple algebraic calculations, one can prove that the coverage probability also reaches the value given in equation (8.2), in the case $p_1 \neq 0$ and $p_2 \neq 0$, when $\Psi\chi^2_{c(n-1)}(1-\alpha/2) < \Psi\chi^2_{c(n-1)}(\alpha/2) < \chi^2_{c(n-1)}(1-p_2) < \chi^2_{c(n-1)}(p_1)$. On the other hand, the coverage probability reaches its minimum value 0, in the situation $\Psi\chi^2_{c(n-1)}(1-\alpha/2) < \chi^2_{c(n-1)}(1-p_2) < \chi^2_{c(n-1)}(p_1) < \Psi\chi^2_{c(n-1)}(\alpha/2)$. We derive below the likelihood ratio based confidence interval for β.

9 Likelihood ratio based confidence interval

The likelihood function based on the two parameter Weibull distribution and the random sample $x_1, \cdots, x_n$ is given by

$$Lw(\theta,\beta) = \beta^n \theta^{-n\beta} \prod_{i=1}^{n} x_i^{\beta-1} \exp -[\sum_{i=1}^{n} (\frac{x_i}{\theta})^{\beta}]. \tag{9.1}$$

For a fixed value β, the maximum likelihood estimator of θ is given by

$$\tilde{\theta} = [\frac{\sum_{i=1}^{n} x_i^{\beta}}{n}]^{\frac{1}{\beta}}. \tag{9.2}$$

Thus the profile likelihood for β is given by

$$Lw(\tilde{\theta},\beta) = \beta^n [\frac{\sum_{i=1}^{n} x_i^{\beta}}{n}]^{-n} \prod_{i=1}^{n} x_i^{\beta-1} \exp[-n], \tag{9.3}$$

and the corresponding profile likelihood ratio is

$$PLR(\beta) = \frac{Lw(\tilde{\theta},\beta)}{Lw(\hat{\theta},\hat{\beta})}, \tag{9.4}$$

where $(\hat{\theta},\hat{\beta})$ denote the maximum likelihood estimate of (θ,β). The $(1-\alpha)100\%$ confidence interval for β is then given by the solution set of the inequation

$$PLR(\beta) > \exp -[\frac{\chi^2_{(1-\alpha,1)}}{2}]. \tag{9.5}$$

10 Simulation results

To check the performance of the conditional interval over its corresponding unconditional one and the likelihood based interval, we performed a simulation study in the cases of $n = 15, 30, 100$, $p = 0.90$, $\alpha = .05, 0.10, 0.20, 0.40, 0.80$, $\Psi = 0.05(0.10)5.05$, $\beta_0 = 10$ and $c = 0.822, 1$. The chosen values for n, α and ψ account for small to large possible values. Without loss of generality, the scale parameter θ has been fixed to unity. We mostly present results for the sample size $n = 30$ since in most studies like reliability the sample size is always small or moderate. To this end, we randomly selected 1000 times random samples of size $n = 15, 30, 100$ and for each sample we carried out the testing of H_0 and the computation of the optimal conditional and unconditional intervals as well as of the likelihood based interval in the case of rejection of H_0. The bounds of the intervals have been computed for all considered c, α and ψ values. The estimation of β and θ by the maximum likelihood method is highly computational. It involves the resolution of two non linear equations. The Newton-Raphson procedure using the starting point for the shape parameter advocated in Zanakis (1979) has been successfully used. Furthermore, the bisection method has been used for solving equations (4.7) and (4.8) in order to find the bounds of the conditional interval.

The results have shown that the parameter α does not affect significantly the properties of the confidence of intervals. Only a slight improvement in the conditional interval has been observed for $\alpha \geq 0.4$. Therefore, we averaged the lengths and the coverage probabilities over all α values. The results have also shown that the conditional interval (CI) maintains almost the same coverage probability as the unconditional interval (UI) except in the vicinity of $\Psi = 1$. This coverage probability is close to the targeted nominal level especially for $c = 1$, see, Figure 2 and Figure 4, in appendix. Figures 2 and 4 also show that the likelihood based interval (LRI) outperforms the others in terms of coverage probability but it overestimates the nominal level. On the other hand, the interval (CI) maintains a much smaller length than (UI) over a large portion of the Ψ values. Confidence intervals (CI) and (UI) have slightly smaller lengths for $c = 1$ but the gap between the two length curves is more significant when c=0.822 as illustrated by Figures 3 and 4. These figures also show that the likelihood based interval has a significantly larger length than the other intervals. Overall, the conditional interval obtained for $c = 0.822$ is better when both length and coverage probability are equally weighted. Furthermore, Figures 5 and 6 show that the empirical distribution of the pivotal statistic (2.1) fits adequately the chi-square distribution with $c(n-1)$ degrees of freedom for both $c = 0.822$ and $c = 1$, especially, for $c = 0.822$. This confirms the theoretical results stated in Bain and Engelhardt (1981). Moreover, we compared the global performance of the intervals (UI) and (LRI) including the non rejection and rejection situation. The results are displayed in the Table 1 below.

The table results show that (LRI) has the largest length and its coverage probability is always greater than the nominal level. The closest coverage probability of (UI) to the nominal level is obtained for $c = 0.822$ but the smallest length is yielded by $c = 1$. The value $c = 0.822$ gives a better combination coverage probability and length for (UI).

Table 1 *Length and coverage probability of the unconditional intervals based on Bain and Engelhardt statistical pivot and likelihood ratio in the case $n = 15, 30, 100$ and $c = 0.822, 1$. LUI, $LLRI$, $CPUI$ and $CPLRI$ represent the lengths and coverage probabilities of (UI) and (LRI), respectively.*

n	c	LUI	$CPUI$	$LLRI$	$CPLRI$
15	.822	7.1852	.9180	7.7108	.9080
15	1.0	6.5291	.88	7.7108	.9080
30	.822	4.8580	.8940	5.2458	.9220
30	1.0	4.4080	.8640	5.2458	.9220
100	.822	2.6062	.9080	2.9294	.9260
100	1.0	2.3636	.8740	2.9294	.9260

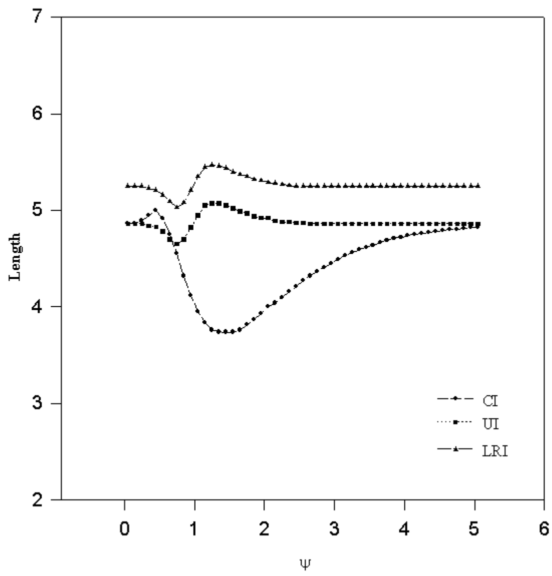

Figure 1 *Length of the confidence intervals for $c = 0.822$ and $n = 30$*

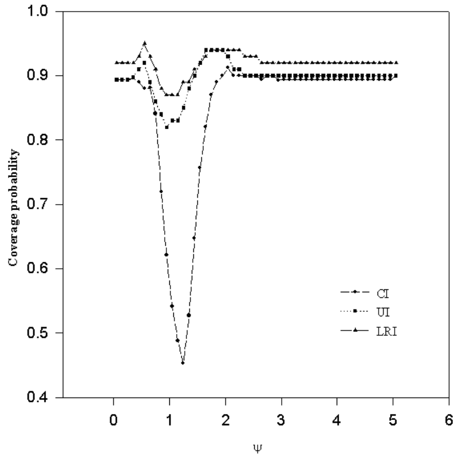

Figure 2 *Coverage probability of the confidence intervals for* $c = 0.822$ *and* $n = 30$

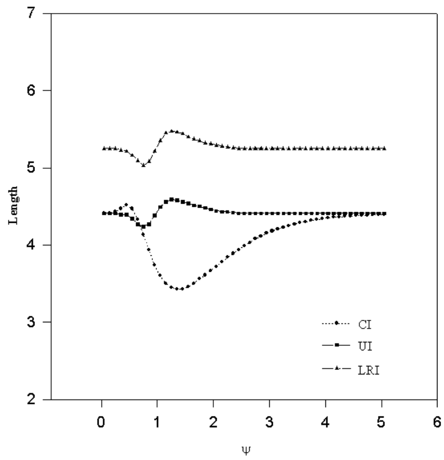

Figure 3 *Length of the confidence intervals for* $c = 1$ *and* $n = 30$

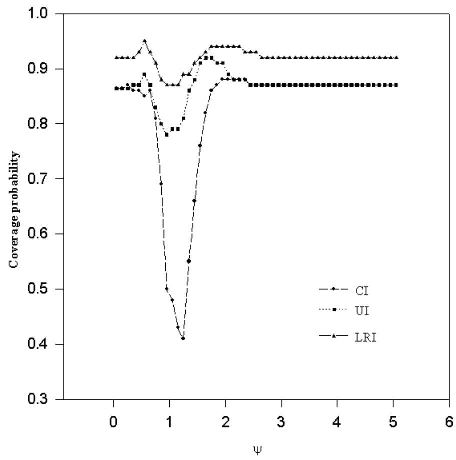

Figure 4 *Coverage probability of the confidence intervals for* $c = 1$ *and* $n = 30$

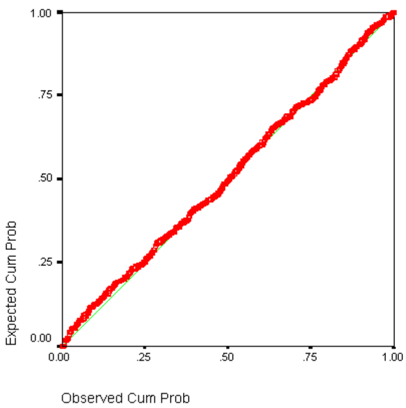

Figure 5 *Chi-square P-P plot of pivot for* $c = 0.822$ *and* $n = 30$

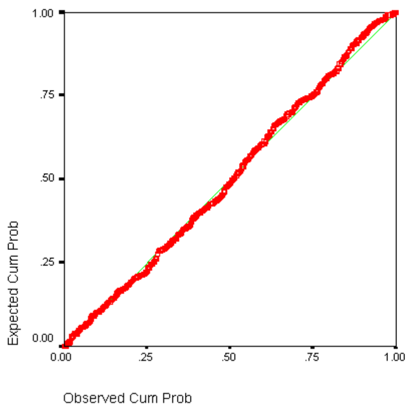

Figure 6 *Chi-square P-P plot of pivot for* $c = 1$ *and* $n = 30$

11 Conclusion

We have investigated here the conditional estimation by confidence interval of the shape parameter β in Weibull model. The confidence interval is constructed only after rejection of preliminary test of significance for the null hypothesis $H_0 : \beta = \beta_0$. The inference is based on the statistical pivot (2.1) with $c = 0.822$ and $c = 1$. Optimal bounds for the conditional confidence interval are obtained following the procedure set forth by Meeks and D'Agostino (1983) and Wardell (1997). This interval is compared in terms of coverage probability and average width to the corresponding optimal unconditional interval. These two intervals are also compared to the likelihood based confidence interval, recommended in Meeker and Escobar (1998). Note that the coverage probability of the unconditional is evaluated under the conditional distribution of the ML estimator of β. As in the finding of Chiou and Han (1994, 1995), the conditional coverage probability of the unconditional interval is not uniformly greater or less than the nominal level. As the values of $\hat{\beta}$ move away from values such that $cn\beta_0^2/\hat{\beta}^2$ is near $\chi^2_{c(n-1)}(\alpha/2)$ or $\chi^2_{c(n-1)}(1-\alpha/2)$ in the critical region associated with the preliminary test, both intervals become similar in terms of average width and coverage probability. However, when $cn\beta_0^2/\hat{\beta}^2$ gets near $\chi^2_{c(n-1)}(\alpha/2)$ or $\chi^2_{c(n-1)}(1-\alpha/2)$, the two intervals behave differently and the conditional interval outperforms the unconditional interval over a significant region of the parameter space, especially, when $\alpha \geq 0.4$ and Ψ or its estimate $\hat{\Psi}$ is at the right vicinity of 1. It is then recommended to use the conditional confidence interval if $\hat{\Psi}$ is not far away from 1, especially from above; this will lead to a significant gain in the estimation accuracy. However, if one has prior knowledge that ψ is far away from $\psi = 1$, then it is better to use the unconditional confidence interval. As in the finding of Chiou and Han (1994, 1995), none of these two intervals totally outperforms the other over the whole parameter space. Therefore, the always use of unconditional confidence intervals independently of the preliminary test outcome can lead to inaccurate estimates. On the hand, the simulation study showed that the likelihood based interval overestimates the confidence nominal level and has a much larger length than the other intervals. We therefore give privilege to the confidence intervals based on Bain and Engelhardt statistic.

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