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On limit theorems for symmetrics of weakly dependent processes

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Abstract: This paper is a generalization of Teicher's (1992) result for an infinite order degenerate U-statistics. The limiting distributions of infinite order degenerate U-statistics under long range dependent process, strictly stationary α and ρ mixing processes have been derived. An r-quick convergence result has also been dealt with.

Key words: Degenerate U-statistics; long range dependent process; r-quick convergence; stationary weakly dependent processes.

1 Introduction and notation

For a double array $\{X_{nj}, 1 \leq j \leq m_n \to \infty, n \geq 1\}$ of random variables on some probability space and any positive integer k, define sums of distinct products of k of the random variables in the n-th row by

$$S_{nk} = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq m_n} X_{n_{i_1}} \ldots X_{n_{i_k}}.$$

Teicher (1992) studied the convergence in law and the almost certain (a.c.) convergence of $S_{nk} \Big/ \left(\sum_{j=1}^{m_n} |X_{nj}|^p\right)^{k/p}$, $0 < p \leq 2$, $k \geq 1$, where the double sequence $\{X_{nj}\}$ of random variables is without any particular probabilistic structure. The purpose of this note is to extend the above result to get invariance principle for an infinite order self-normalized generalized U-statistic which can be used to test simultaneous independence of a given set of observations. The result due to Teicher (1992) can be obtained from our expression.

Our method of proof is quite elementary and is different from that of Teicher (1992). This approach unifies many of the known results in the literature of dependent variables.

Mandelbaum and Taqqu (1984) derived the invariance principle of an infinite order symmetric statistic. Later on Mandrekar and Rao (1989) and then Feldman and Rachev (1993) made their contributions in this direction.

Our result for convergence in law has been applied for an array of martingale difference sequences (see Teicher, 1992), i.i.d. random variables in the domain of attraction of a stable law and a sequence of exchangeable random variables. Here

the applications mainly include the study of strictly stationary strong mixing and ρ-mixing processes and long range dependent process. For finite as well as infinite variance cases the limiting distribution of the symmetric statistic, under different mixing setup, has been expressed as an infinite sum of Hermite polynomials in Wiener process. But it is interesting to note that, under long range dependent setup, the limiting distribution of the symmetric statistic has been expressed as $\sum_{k=0}^{\infty} B_H^k(t)$, where $B_H(t)$ is a fractional Brownian motion with parameter $1/2 < H < 1$. Our result can also be applied for centered strictly stationary random fields satisfying $\rho^*(s) \to 0$ as $s \to \infty$, ρ^* being the "unrestricted ρ-mixing coefficient" (for definition and for example Peliagrad, 1998). An r-quick version of Marcinkiewicz-Zygmund strong law has been dealt with for strictly stationary mixing sequence whose bivariate tail probabilities decrease faster than the univariate tail probability (for $k = 1$ see Lai, 1977). Sometimes we use the Vinogradov symbol $\ll$ to indicate an inequality containing some unspecified positive constant factor.

2 Main results

Define

$$\begin{aligned} S_{n0}^t &= 1, \\ S_{nk}^t &= \sum_{1 \le i_1 < \cdots < i_k \le [m_n t]} X_{n_{i_1}} \ldots X_{n_{i_k}}, \quad 0 \le t \le 1, \ k \ge 1, \\ S_{nk} &= S_{nk}^1 \ \text{ for } k \ge 0, \\ W_{np} &= \left(\sum_{j=1}^{m_n} |X_{nj}|^p \right)^{1/p}, \ p > 0 \end{aligned}$$

and

$$X_n^* = \max_{1 \le j \le m_n} |X_{nj}|.$$

Let

$$Y_n^t = \sum_{k=0}^{m_n} (S_{nk}^t / W_{np}^k).$$

Then the following theorems hold.

Theorem 1 *For $p = 2$ let*

$$\frac{S_{n1}^t}{W_{n2}} \xrightarrow{\mathcal{D}} Z_F^t, \tag{2.1}$$

$$\frac{\sum_{j=1}^{[m_n t]} |X_{nj}|^2}{W_{n2}^2} \xrightarrow{P} t \ \textit{as } n \to \infty, \ \textit{uniformly in } t, 0 \le t \le 1, \tag{2.2}$$

and

$$\frac{X_n^*}{W_{n2}} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2.3}$$

where Z_F^t is a random process with marginal distribution F (when $t = 1$) and $\xrightarrow{\mathcal{D}}$ denotes convergence in Skorohord topology.

Then we have for $p = 2$

$$Y_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(Z_F^t) \text{ as } n \to \infty,$$

where $H_k(\cdot)$ is the Hermite polynomial of degree k.

Theorem 2 *For $p \in (0, 2)$ let c_{np} be some constant satisfying*

$$c_{np} \frac{S_{n1}}{W_{np}} \xrightarrow{d} Z_F \text{ as } n \to \infty, \tag{2.4}$$

$$c_{np} \frac{|X_{nj}|}{W_{np}} < 1 \text{ a.s. for } 1 \le j \le m_n, \text{ for large } n \tag{2.5}$$

and

$$c_{np} \frac{W_{n2}}{W_{np}} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{2.6}$$

Then we have

$$Z_n = \sum_{k=0}^{m_n} c_{np}^k (S_{nk}/W_{np}^k) \xrightarrow{d} \sum_{k=0}^{\infty} \frac{1}{k!} Z_F^k \text{ as } n \to \infty.$$

Now we define

$$\begin{aligned} \tilde{Y}_n^t &= \sum_{k=0}^{n} (\tilde{S}_{nk}^t / \tilde{W}_{np}^k), \ 0 \le t \le 1, \\ \tilde{Y}_n &= \tilde{Y}_n^1, \\ \tilde{Z}_n &= \sum_{k=0}^{n} c_{np}^k (\tilde{S}_{nk} / \tilde{W}_{np}^k), \end{aligned}$$

where

$$\begin{aligned} \tilde{S}_{n0}^t &= 1, \\ \tilde{S}_{nk}^t &= \sum_{1 \le i_1 < \cdots < i_k \le [nt]} X_{i_1} \ldots X_{i_k}, \quad k \ge 1, \\ \tilde{S}_{nk} &= \tilde{S}_{nk}^1, \quad k \ge 0, \end{aligned}$$

and

$$\tilde{W}_{np} = \left(\sum_{i=1}^{n} |X_i|^p \right)^{1/p}, \ p > 0.$$

Corollary 1 *Let $\{X_{nj} = X_j,\ 1 \le j \le m_n\}$, where $\{X_j,\ j > 1\}$ is a sequence of i.i.d. random variables in the domain of attraction of a stable law with characteristic exponent $\alpha \in (1,2)$ and $EX_1 \neq 0$. Then for $p \in (\alpha, 2)$*

$$\tilde{Z}_n \xrightarrow{d} \sum_{k=0}^{\infty} \frac{1}{k!} Z_F^k \text{ as } n \to \infty,$$

where Z_F is defined as in Theorem 1, $c_{np} = \dfrac{L(1/n)}{n^{1-1/\alpha}\tilde{W}_{np}EX_1}$ and the distribution F and slowly varying function L are specified in Csörgo and Horva'th (1988).

Corollary 2 *Let $\{X_n\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$ and $E(|X_1|^2 \log\log|X_1|) < \infty$. Then for $p = 2$*

$$\tilde{Y}_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(W_t) \text{ as } n \to \infty,$$

where W_t is a standard Brownian motion with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = s \wedge t$.

Corollary 3 *Let $\{X_n\}$ be a sequence of exchangeable random variables with $EX_1 = 0$, $E(|X_1|^2 \log\log|X_1|) < \infty$.*

Then for $p = 2$

$$\tilde{Y}_n \xrightarrow{d} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(Z) \text{ as } n \to \infty,$$

where Z is a standard Normal variate.

Corollary 4 *Let $\{X_n\}$ be a centered strictly stationary ρ-mixing sequence of random variables. Let $g : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that for some $0 < \delta < 1$, $x^\delta/g(x)$ is a non-decreasing function. Also suppose that*

$$EX_1^4 g(|X_1|^2) < \infty, \tag{2.7}$$

$$\exp\left((2+\varepsilon^*) \sum_{k=1}^{[\log(n)]} \rho(2^k)\right) = O(g(n^{1/2})) \text{ for some } 0 < \varepsilon^* < 1, \tag{2.8}$$

and

$$\sigma_n^2/n \to \sigma^2 \text{ as } n \to \infty, \tag{2.9}$$

where

$$\sigma_n^2 = E\left(\sum_{k=1}^{n} X_k\right)^2.$$

Then for $p = 2$

$$\tilde{Y}_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(W_t) \text{ as } n \to \infty,$$

where W_t is a Brownian motion with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = (s \wedge t)\sigma^2/\sigma_0^2$, $\sigma_0^2 = EX_1^2$.

Let $\{X_n\}$ be a sequence of centered strictly stationary strong-mixing sequence of random variables with mixing coefficient α_n. Define the strong-mixing function $\alpha(t)$ by

$$\alpha(t) = \alpha_{[t]}.$$

Denote by Q the quantile function of $|X_1|$ which is the inverse function of

$$t \to P(|X| > t).$$

Assume that

$$\int_0^1 \alpha^{-1}(u)du < \infty \tag{2.10}$$

and

$$\int_0^1 \alpha^{-1}(u)[Q(u)]^4 du < \infty, \tag{2.11}$$

where α^{-1} is the inverse function of α defined by

$$\alpha^{-1}(u) = \inf\{t : \alpha(t) \le u\}.$$

Note that

$$\int_0^1 \alpha^{-1}(u)[Q(u)]^2 du \le \left(\int_0^1 \alpha^{-1}(u)[Q(u)]^4 du\right)\left(\int_0^1 \alpha^{-1}(u)du\right)$$

and hence the above two conditions imply that

$$\int_0^1 \alpha^{-1}(u)[Q(u)]^2 du < \infty. \tag{2.12}$$

Then the following result holds.

Corollary 5 *For $p = 2$*

$$\tilde{Y}_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(W_t) \ \textit{as } n \to \infty,$$

where W_t is a Brownian motion with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = (s \wedge t)\sigma^2/\sigma_0^2$, $\sigma_0^2 = EX_1^2$ and $\sigma^2 = \sum_{n=1}^{\infty} \operatorname{cov}(X_1, X_n)$.

Let $\{X_n\}$ be a centered strictly stationary sequence of random variables and Z be the set of all integers. For $T \subset Z$ and $S \subset Z$ define

$$\rho(\mathcal{M}_T, \mathcal{M}_S) = \sup_{f \in L_2(M_T), g \in L_2(M_S)} |\operatorname{corr}(f, g)|,$$

where

$$\mathcal{M}_T = \sigma\langle X_i : i \in T\rangle \text{ and } \mathcal{M}_S = \sigma\langle X_i : i \in S\rangle.$$

Also define

$$\rho^*(n) = \sup \rho(\mathcal{M}_T, \mathcal{M}_S),$$

where the supremum is taken over all nonempty, disjoint subsets S and T of Z satisfying $dist(S,T) \geq n$.

Assume that the sequence of random variables $\{X_n\}$ satisfies

$$EX_1^4 < \infty \tag{2.13}$$

and

$$\rho^*(n) \to 0 \text{ as } n \to \infty. \tag{2.14}$$

According to Bradley (1993), $\rho^*(n) \to 0$ as $n \to \infty$ implies $\alpha(n) \to 0$ as $n \to \infty$, where $\alpha(n)$ is the strong mixing coefficient. Also assume that

$$\sigma_n^2 \to \infty \text{ as } n \to \infty. \tag{2.15}$$

Then the following corollary of Theorem 1 holds.

Corollary 6 *For $p = 2$*

$$\tilde{Y}_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(W_t),$$

where W_t is a Brownian motion with $EW_t = 0$, $\operatorname{cov}(W_s, W_t) = (s \wedge t)\sigma^2/\sigma_0^2$ and $\lim_{n\to\infty} \sigma_n^2/n = \sigma^2 > 0$ and $\sigma_0^2 = EX_1^2$.

Let S and T be two subsets of the set of all integers. For any two finite subsets $S^* \subseteq S$ and $T^* \subseteq T$ define two random variables V and W such that

$$V = \sum_{k \in S^*} a_k X_k \quad \text{and} \quad W = \sum_{k \in T^*} b_k X_k,$$

where a_k's and b_k's are real numbers. Define

$$r(S,T) = \sup_{S^* \subseteq S, T^* \subseteq T} \frac{\operatorname{cov}(V,W)}{\|V\|_2 \|W\|_2},$$

where the supremum is taken over all pairs of random variables V and W. For every real number $s \geq 1$ define

$$r^*(s) = \sup r(S,T),$$

where the supremum is taken over all pairs of nonempty disjoint subsets S and T of Z such that $\operatorname{dist}(S,T) \geq s$ (Bradley, 1992, pp. 356). Define

$$r^* = r^*(1).$$

The following result will be used to prove Corollary 7.

Let $\{X_n\}$ be a sequence of weakly stationary random variables. Then the following two statements are equivalent:

(i) $r^*(1) < 1$ and $r^*(s) \to 0$ as $s \to \infty$,

(ii) X has a continuous positive spectral density on $T' \subset Z$, T' being of unit length on real line (Bradley, 1992).

Again we define $r_n^*(s)$ replacing X_i by $X_i^{(n)}$ in $r^*(s)$ where

$$X_i^{(n)} = X_i I(|X_i| \le t_n) - EX_1 I(|X_1| \le t_n).$$

It is easy to see that

$$r_n^* \to r^* \text{ as } n \to \infty. \tag{2.16}$$

Now we state the following result.

Corollary 7 *Let $\{X_n\}$ be a strictly stationary ϕ-mixing sequence of random variables with $\phi(1) < 1$. Assume that X_1 has a continuous positive spectral density and*

$$H(a) = EX_1^2 I(|X_1| \le a) \tag{2.17}$$

is slowly varying as $a \to \infty$.

Then for $p = 2$ we get

$$\tilde{Y}_n^t \xrightarrow{\mathcal{D}} \sum_{k=0}^{\infty} \frac{1}{k!} H_k(W_t) \text{ as } n \to \infty.$$

Here W_t is a Gaussian process with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = \frac{2}{\pi}((1+c)/(1-c))(s \wedge t)$, where $-r^* \le c \le r^*$.

Now we shall study another closely related and interesting result for stationary process in which the limiting distribution does not involve Hermite polynomials.

Theorem 3 *Let $\{X_n\}$ be a centered strictly stationary Gaussian process with covariance function $r(k) = EX_1X_{k+1}$, $r(k) = k^{-D}L(k)$, $0 < D < 1$, where L is slowly varying at infinity.*

Define

$$\hat{Y}_n^t = \sum_{k=0}^{n} c_n^k \tilde{S}_{nk}^t / \tilde{W}_{n2}^k,$$

where $c_n = n^{1/2}\sigma_0/d_n$, $\sigma_0^2 = \operatorname{var}(X_1)$, $d_n \sim n^H L(n)$, $\frac{1}{2} < H = 1 - D/2 < 1$ and $L(n)$ is a slowly varying function at infinity. Note that d_n can be taken to be $\operatorname{var}\left(\sum_{i=1}^n X_i\right)$ *for large n (see Taqqu, 1975). Then*

$$\hat{Y}_n^t \xrightarrow{\mathcal{D}} \frac{1}{k!} \sum_{k=0}^{\infty} B_H^k(t),$$

where $B_H(t)$ is a fractional Brownian motion.

The following theorem is a generalization of the almost sure version of Teicher's (1992) result for infinite order U-statistics.

Theorem 4 *Let c_{np} be constants with $c_{n2} \equiv 1$. If $\{Z_F, X_{nj}, 1 \le j \le m_n \to \infty\}$ is an array of random variables on a probability space satisfying for some p in $(0, 2]$*

$$c_{np}\frac{S_{n1}}{W_{np}} \xrightarrow{a.s.} Z_F \text{ as } n \to \infty, \tag{2.18}$$

$$c_{np}\frac{|X_{nj}|}{W_{np}} < 1 \text{ a.s. for } 1 \le j \le m_n, \text{ for large } n, \tag{2.19}$$

and

$$\frac{X_n^*}{W_{np}} \xrightarrow{a.s.} 0 \quad \text{or} \quad c_{np}\frac{W_{n2}}{W_{np}} \xrightarrow{a.s.} 0 \text{ as } n \to \infty \tag{2.20}$$

according as $p = 2$ or $p < 2$. Then as $n \to \infty$

$$Z_n \xrightarrow{a.s.} \begin{cases} \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} H_k(Z_F), & \text{for } p = 2, \\ \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} Z_F^k, & \text{for } 0 < p < 2. \end{cases}$$

We say that a sequence Z_n of random variables converges to 0 r-quickly for some $r > 0$ if

$$E(\sup\{n \ge 1 : |Z_n| \ge \varepsilon\})^r < \infty \text{ for all } \varepsilon > 0 \ (\sup \phi = 0).$$

Obviously $Z_n \to 0$ r-quickly for some $r > 0$ implies that $Z_n \to 0$ a.s..

Next the following closely related result which in turn implies r-quickly version of Marcinkiewicz-Zygmund strong law of large number for U-statistics will be dealt with.

Theorem 5 *Let $\{X_n\}$ be strictly stationary sequence of random variables such that there exists $\beta > 1$ and a positive integer m for which*

$$\sup_{i>m} P[|X_1| > x,\ |X_i| > x] = O(P^{\beta}[|X_1| > x]) \tag{2.21}$$

holds as $x \to \infty$.

Also suppose that $E|X_1|^p < \infty$, for some $p > 0$ and choose α such that

$$p\alpha \ge \max\{(\beta p + 1), \beta/(\beta - 1)\}.$$

Define

$$\hat{Z}_n = \sum_{k=0}^{n} (\tilde{S}_{nk}/n^{\alpha k}).$$

Then as $n \to \infty$ we get $\hat{Z}_n \to 1$ $(p\alpha - 1)$-quickly.

3 Proof of main theorems

Proof of Theorem 1 Write

$$\begin{aligned} Y_n^t &= \sum_{k=0}^{m_n} \sum_{1\le i_1<\cdots<i_k\le[m_n t]} (X_{ni_1}\dots X_{ni_k}/W_{n2}^k) \\ &= \prod_{j=1}^{[m_n t]} (1+X_{nj}/W_{n2}). \end{aligned} \tag{3.1}$$

Since $\dfrac{|X_{nj}|}{W_{n2}} < 1$ for $1 \le j \le m_n$ and for large n, taking logarithm on both sides of (3.1) we get

$$\log Y_n^t = \sum_{j=1}^{[m_n t]} (X_{nj}/W_{n2}) - \frac{1}{2}\sum_{j=1}^{[m_n t]} (X_{nj}^2/W_{n2}^2) + \varepsilon_n(t), \tag{3.2}$$

where

$$|\varepsilon_n(t)| \le \sum_{j=1}^{[m_n t]} |X_{nj}|^3/W_{n2}^3 \le (X_n^*/W_{n2}).$$

Hence by (2.3) it follows that

$$\varepsilon_n(t) \xrightarrow{P} 0 \text{ as } n\to\infty \text{ uniformly in } t, 0\le t\le 1.$$

Thus by (2.1), (2.2) and (3.2) we have

$$\log Y_n^t \xrightarrow{\mathcal{D}} Z_F^t - \frac{1}{2}t \text{ as } n\to\infty,$$

i.e.

$$Y_n^t \xrightarrow{\mathcal{D}} \exp\left(Z_F^t - \frac{1}{2}t\right) \text{ as } n\to\infty$$

and the result follows.

Proof of Theorem 2 Write

$$\begin{aligned} Z_n &= \sum_{k=0}^{m_n} \sum_{1\le i_1<\cdots<i_k\le m_n} c_{np}^k (X_{ni_1}\dots X_{ni_k}/W_{np}^k) \\ &= \prod_{j=1}^{m_n} \left(1 + c_{np}\frac{X_{nj}}{W_{np}}\right). \end{aligned} \tag{3.3}$$

Since (2.5) entails $c_{np}\dfrac{|X_{nj}|}{W_{np}} < 1$ a.s. for $1 \le j \le m_n$ taking logarithm on both sides of (3.3) we get

$$\log Z_n = c_{np}\frac{\sum_{j=1}^{m_n} X_{nj}}{W_{np}} + \varepsilon_n, \tag{3.4}$$

where

$$|\varepsilon_n| \leq \sum_{j=1}^{m_n} c_{np}^2(|X_{nj}|^2/W_{np}^2) \leq (c_{np}W_{n2}/W_{np})^2. \tag{3.5}$$

Hence by (2.6) it follows that

$$\varepsilon_n \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Thus by (3.4) and (2.4) we get

$$\log Z_n \xrightarrow{d} Z_F \text{ as } n \to \infty,$$

i.e.

$$Z_p \xrightarrow{d} \exp(Z_F) \text{ as } n \to \infty$$

and the result follows.

Proof of Theorem 3 Write

$$\hat{Y}_n^t = \prod_{j=1}^{[nt]} \left(1 + \frac{c_n X_j}{\tilde{W}_{n2}}\right). \tag{3.6}$$

Since $\dfrac{|X_j|}{\tilde{W}_{n2}} < 1$ for $1 \leq j \leq n$ and for large n and since $c_n = o(1)$ we have for large n $\dfrac{c_n|X_j|}{\tilde{W}_{n2}} < 1$ for $1 \leq j \leq n$.

Taking logarithm on both sides of (3.6) we get

$$\log \hat{Y}_n^t = \sum_{j=1}^{[nt]} c_n \frac{X_j}{\tilde{W}_{n2}} + \varepsilon_n(t), \tag{3.7}$$

where

$$|\varepsilon_n(t)| \leq \sum_{j=1}^{n} c_n^2 X_j^2/\tilde{W}_{n2}^2 \leq c_n^2 \to 0 \text{ as } n \to \infty. \tag{3.8}$$

Since $\dfrac{1}{d_n}\displaystyle\sum_{j=1}^{[nt]} X_j \xrightarrow{\mathcal{D}} B_H(t)$ as $n \to \infty$ (see Taqqu, 1975) the theorem follows from ergodicity, (3.7) and (3.8).

Proof of Theorem 4 The proof is similar to that of Theorems 1 and 2.

Proof of Theorem 5 Write

$$\hat{Z}_n = \prod_{i=1}^{n} \left(1 + \frac{X_i}{n^\alpha}\right). \tag{3.9}$$

Let us observe that since $\alpha p \geq \beta/(\beta-1) > 1$, for any $\varepsilon > 0$ we have

$$\sum_{i=1}^{\infty} P(|X_t| > \varepsilon i^{\alpha}) \leq \frac{E|X_1|^p}{\varepsilon^p} \sum_{i=1}^{\infty} \frac{1}{i^{\alpha p}} < \infty.$$

Hence by Borel-Cantelli lemma

$$|X_i| \leq \varepsilon i^{\alpha} \leq \varepsilon n^{\alpha} \text{ a.s. for } n \geq i \geq \text{ some } N,\ N < \infty.$$

Since

$$\lim_n \hat{Z}_n = \lim_n \prod_{i=N}^{n} \left(1 + \frac{X_i}{n^{\alpha}}\right),$$

without loss of generality, we can assume $\dfrac{|X_i|}{n^{\alpha}} < 1$ a.s. for all $i \leq n$.

Taking logarithm we get from (3.9)

$$\log \hat{Z}_n = \sum_{i=1}^{n} X_i/n^{\alpha} + \varepsilon_n, \tag{3.10}$$

where

$$|\varepsilon_n| \leq \sum_{i=1}^{n} \frac{X_i^2}{n^{2\alpha}} \leq \left(\max_{1\leq i\leq n} \frac{|X_i|}{n^{\alpha}}\right) \sum_{i=1}^{n} \frac{|X_i|}{n^{\alpha}} < \sum_{i=1}^{n} \frac{|X_i|}{n^{\alpha}} \quad \text{a.s.}$$

and hence by (3.10)

$$|\log \hat{Z}_n| \leq 2 \sum_{i=1}^{n} \frac{|X_i|}{n^{\alpha}} \quad \text{a.s.} \tag{3.11}$$

By Theorem 1.(i) of Lai (1977) we have

$$\sum_{i=1}^{n} \frac{|X_i|}{n^{\alpha}} \to 0 \quad (p\alpha - 1)\text{-quickly as } n \to \infty$$

and hence by (3.11) the result follows.

4 Proof of corollaries

Proof of Corollary 1 Since $c_{np} = o(1)$ and $\dfrac{\tilde{W}_{n2}}{\tilde{W}_{np}} \leq 1$, $p \leq 2$, (2.5) and (2.6) hold immediately. Also (2.4) holds by Csörgo and Horv'ath (1988) and hence the result follows from Theorem 2.

Proof of Corollary 2 From Donsker's theorem and SLLN it follows that the limit of $S_{n1}^t/\tilde{W}_{n2}$ is a Brownian motion with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = s \wedge t$. Consequently (2.1) holds.

Let $D[0,1]$ be the space of right continuous functions on $[0,1]$ endowed with Skorohord topology ρ. Write

$$\begin{aligned}\rho\left(\frac{\sum_{i=1}^{[nt]} X_i^2}{\sum_{i=1}^{n} X_i^2}, t\right) &\leq \rho\left(\frac{\sum_{i=1}^{[nt]} X_i^2/n}{\sum_{i=1}^{n} X_i^2/n}, \frac{tEX_1^2}{\sum_{i=1}^{n} X_i^2/n}\right) + \rho\left(\frac{tEX_1^2}{\sum_{i=1}^{n} X_i^2/n}, t\right) \\ &= I + II, \text{ say.} \end{aligned} \tag{4.1}$$

By WLLN

$$II \leq \sup_{0\leq t\leq 1}\left|t\left(\frac{EX_1^2}{\sum_{i=1}^{n} X_i^2/n} - 1\right)\right| \leq \left|\frac{EX_1^2}{\sum_{i=1}^{n} X_i^2} - 1\right| \xrightarrow{P} 0 \text{ as } n\to\infty. \tag{4.2}$$

By WLLN and maximal inequality

$$\begin{aligned}&\sup_{0\leq t\leq 1}\left|\frac{\sum_{i=1}^{[nt]}(X_i^2 - EX_1^2)/n}{\sum_{i=1}^{n} X_i^2/n}\right| \\ &= \frac{1}{\sum_{i=1}^{n} X_i^2/n}\sup_{0\leq t\leq 1}\left|\sum_{i=1}^{[nt]}(X_i^2 - EX_1^2)/n\right| \xrightarrow{P} 0 \text{ as } n\to\infty \end{aligned} \tag{4.3}$$

(see Mandrekar and Rao, 1989, pp. 274).

Thus by (4.1), (4.2) and (4.3) we have

$$\rho\left(\frac{\sum_{i=1}^{[nt]} X_i^2}{\sum_{i=1}^{n} X_i^2}, t\right) \xrightarrow{P} 0 \text{ as } n\to\infty$$

and hence (2.2) holds.

Now

$$P(X_n^* > \varepsilon\sqrt{n}) \leq \sum_{i=1}^{n} P(|X_i| > \varepsilon\sqrt{n}) \leq \frac{E(X_1^2 \log\log|X_1|)}{\varepsilon^2(\log\log(\varepsilon\sqrt{n}))} \to 0 \text{ as } n\to\infty.$$

Hence by SLLN (2.3) holds and the corollary follows from Theorem 1.

Proof of Corollary 3 Define $\tau = \bigcap_{n=1}^{\infty}\sigma\{X_n, X_{n+1}, \ldots\}$. Since

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i \xrightarrow{d} N(0, E(X_1^2|\tau)) \text{ as } n\to\infty$$

and following Kingman (1978) we get

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{a.s.}} E(X_1^2|\tau) \text{ as } n\to\infty,$$

it follows that

$$\frac{\tilde{S}_{n1}}{\tilde{W}_{n2}} \xrightarrow{d} Z \text{ as } n \to \infty$$

and hence (2.1) holds.

Also since $E(X_1^2 \log\log|X_1|) < \infty$, (2.3) holds by Kingman (1978) and the corollary follows from Theorem 1.

Proof of Corollary 4 By Theorem 4.2.1 of Zhengyan and Chuanrong (1996) and ergodicity

$$\frac{\tilde{S}_{n1}^t}{\tilde{W}_{n2}} \xrightarrow{\mathcal{D}} W_t \text{ as } n \to \infty$$

and hence (2.1) holds.

By (2.7) we have $P(X_n^* > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$ for arbitrary $\varepsilon > 0$ and hence (2.3) follows from ergodicity.

Now note that following the proof of Theorem 4.2.1 of Zhengyan and Chuanrong (1996) we have for arbitrary $\varepsilon > 0$

$$P\left(\sup_{0\le t\le 1}\left|\sum_{j=1}^{[nt]}(X_j^2 - EX_1^2)\right| > \varepsilon\sigma_n^2\right) \to 0 \text{ as } n \to \infty. \tag{4.4}$$

Thus (2.2) holds by ergodicity and (4.4) and hence the corollary follows from Theorem 1.

Proof of Corollary 5 Since (2.12) holds, by Theorem 1 of Doukhan, Massart and Rio (1994) and ergodicity we have

$$\frac{\tilde{S}_{n1}^t}{\tilde{W}_{n2}} \xrightarrow{\mathcal{D}} W_t \text{ as } n \to \infty$$

and hence (2.1) holds.

Note that since $\alpha^{-1}(u) \ge 1$ condition (2.11) ensures that $\int_0^1 [Q(u)]^4 du < \infty$, i.e.

$$E|X_1|^4 < \infty\ . \tag{4.5}$$

Thus by (4.5) we have $P(X_n^* > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$ and hence (2.3) holds by ergodicity.

Now write $X_i' = X_i^2 - EX_1^2$, $\{X_n'\}$ being centered strictly stationary strong mixing process with mixing coefficient α_n, by (2.11) and following the proof of Theorem 1 of Doukhan, Massart and Rio (1994) we have

$$P\left(\sup_{0\le t\le 1}\left|\sum_{i=1}^{[nt]} X_i'\right| > \varepsilon n\right) \to 0 \text{ as } n \to \infty\ . \tag{4.6}$$

Thus (2.2) follows from (4.6) and ergodicity. Consequently the corollary holds from Theorem 1.

Proof of Corollary 6 By Corollary 2.1 of Peligrad (1998) and ergodicity

$$\frac{\tilde{S}_{n1}^t}{\tilde{W}_{n2}} \xrightarrow{\mathcal{D}} W_t \text{ as } n \to \infty$$

and hence (2.1) holds.

By (2.13) for arbitrary $\varepsilon > 0$ we have $P(X_n^* > \varepsilon\sqrt{n}) \to 0$ as $n \to \infty$ and hence (2.3) follows from ergodicity.

Write

$$S_n'(t) = \sum_{i=1}^{[nt]} (X_i^2 - EX_1^2), \quad 0 \le t \le 1.$$

By (2.13), (2.14) and Proposition 2.1 of Peligrad (1998), $\sup_{0\le t\le 1} S_n'^2(t)/n$ is uniformly integrable and hence for arbitrary $\varepsilon > 0$,

$$P\left(\sup_{0\le t\le 1} |S_n'(t)| > \varepsilon n\right) \to 0 \text{ as } n \to \infty\ . \tag{4.7}$$

Thus (2.2) follows from ergodicity and (4.7) and hence the corollary follows from Theorem 1.

For the proof of the next corollary of Theorem 1 we state the following result which can be derived from equations (3.2) and (3.3), Lemma 3.1 of Bradley (1988) and Lemma 7 of Bradley (1992).

Lemma 1 *(A variate of lemmas of Bradley, 1988 and 1992.) Defining*

$$t_n = \sup\left\{t > 0 : t^{-2}H(t) \ge \frac{1}{n}\right\}$$

we have

$$t_n^2 = nH(t_n) \text{ as } n \to \infty, \tag{4.8}$$
$$t_n \to \infty \text{ monotonically}, \tag{4.9}$$
$$nP(|X_1| > t_n) \to 0 \text{ as } n \to \infty, \tag{4.10}$$
$$nE(|X_1|I(|X_1| > t_n) = o(t_n) \text{ as } n \to \infty, \tag{4.11}$$
$$E|X_1|^4 I(|X_1| \le t_n) = o(nH^2(t_n)) \text{ as } n \to \infty. \tag{4.12}$$

Proof of Corollary 7 We shall show that

$$\sum_{i=1}^{n} \frac{X_i^2}{t_n^2} \xrightarrow{P} 1 \text{ as } n \to \infty. \tag{4.13}$$

Now note that

$$\frac{1}{t_n^2}\sum_{i=1}^{n} X_i^2 I(|X_i| > t_n) \leq \left(\frac{1}{t_n}\sum_{i=1}^{n} |X_i| I(|X_i| > t_n)\right)\left(\frac{X_n^*}{t_n}\right). \tag{4.14}$$

By (4.10) and (4.11) it follows from (4.14) that

$$\frac{t}{t_n^2}\sum_{i=1}^{n} X_i^2 I(|X_i| > t_n) \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{4.15}$$

Hence it is sufficient to prove that

$$\frac{\sum_{i=1}^{n} X_i^2 I(|X_i| \leq t_n)}{t_n^2} \xrightarrow{P} 1 \text{ as } n \to \infty. \tag{4.16}$$

Define

$$Y_i = X_i^2 I(|X_i| \leq t_n) - EX_i^2 I(|X_i| \leq t_n)$$

and

$$S^{(n)} = \sum_{i=1}^{n} Y_i.$$

By Lemma 1 of Bradley (1992) and (4.12) it can be shown that

$$o(t_n^4) \ll E|S^{(n)}|^2 \ll o(t_n^4). \tag{4.17}$$

Now by Central Limit Theorem for ϕ-mixing process we have

$$E|S^{(n)}| \sim \left|\frac{\pi}{2}\right|^{-1/2} \sqrt{E|S^{(n)}|^2}.$$

Hence by (4.17) we get

$$o(t_n^2) \ll E|S^{(n)}| \ll o(t_n^2) \text{ as } n \to \infty. \tag{4.18}$$

Thus we have

$$\frac{|S^{(n)}|}{t_n^2} \xrightarrow{P} 0 \text{ as } n \to \infty$$

and hence (4.16) and consequently (4.13) hold.

Now write

$$b_n = E\left|\sum_{i=1}^{n} X_i\right|.$$

We want to show that

$$\frac{b_n}{t_n} \to \sqrt{\frac{1+c}{1-c}\left(\frac{2}{\pi}\right)} \text{ as } n \to \infty, \tag{4.19}$$

where $-r^* \le c \le r^*$.

Note that

$$\begin{aligned}
&E\left|\sum_{i=1}^{n}\{X_iI(|X_i|\le t_n)-E[X_iI(|X_i|\le t_n)]\}\right| \\
&\quad -E\left|\sum_{i=1}^{n}\{X_iI(|X_i|> t_n)-E[X_iI(|X_i|> t_n)]\}\right| \le b_n \\
&\le E\left|\sum_{i=1}^{n}\{X_iI(|X_i|\le t_n)-E[X_iI(|X_i|\le t_n)]\}\right| \\
&\quad +E\left|\sum_{i=1}^{n}\{X_iI(|X_i|> t_n)-E[X_iI(|X_i|> t_n)]\}\right| . \qquad (4.20)
\end{aligned}$$

By stationarity and (4.11) we have

$$E\left|\sum_{i=1}^{n}\{X_iI(|X_i|\ge t_n)\}\right| = o(t_n) \text{ as } n\to\infty. \qquad (4.21)$$

By Lemma 1 of Bradley (1992) and Central Limit Theorem for ϕ-mixing process we get for large n

$$\begin{aligned}
&\left(\frac{1-r_n^*}{1+r_n^*}\right)\left[nH(t_n)-nE[X_1I(|X_1|\le t_n)]^2\right] \\
&\le \frac{\pi}{2}\left[E\left|\sum_{i=1}^{n}\{X_iI(|X_i|\le t_n)-E[X_1I(|X_1|\le t_n)]\}\right|\right]^2 \\
&\le \left(\frac{1+r_n^*}{1-r_n^*}\right)\left[nH(t_n)-n(E[X_1I(|X_1|\le t_n)])^2\right].
\end{aligned}$$

Hence

$$\frac{1}{t_n}E\left|\sum_{i=1}^{n}\{X_iI(|X_i|\le t_n)-E[X_1I(|X_1|\le t_n)]\}\right| \to \sqrt{\frac{1+c}{1-c}\left(\frac{2}{\pi}\right)} \text{ as } n\to\infty, \qquad (4.22)$$

where $-r^* \le c \le r^*$.

Thus (4.19) holds from (4.20), (4.21) and (4.22).

Now by (4.13), (4.19) and Theorem 2.1 due to Peligrad (1990) we get

$$\frac{\tilde{S}_n^t}{\tilde{W}_{n2}} = \frac{\left(\frac{\tilde{S}_n^t}{b_n}\right)}{\left(\frac{t_n}{b_n}\right)\left(\frac{\tilde{W}_{n2}}{t_n}\right)} \xrightarrow{\mathcal{D}} W_t \text{ as } n\to\infty.$$

W_t is a Gaussian process with $EW_t = 0$ and $\operatorname{cov}(W_s, W_t) = \left(\frac{2}{\pi}\right)\left(\frac{1+c}{1-c}\right)(s\wedge t)$, where $-r^* \le c \le r^*$ and hence (2.1) holds.

By Proposition 3.2 of Peligrad (1990) we get

$$P\left(\sup_{0\le t\le 1}\left|\sum_{i=1}^{[nt]}[X_i^2 I(|X_i|\le t_n)-E(X_i^2 I(|X_i|\le t_n))]\right|>\varepsilon t_n^2\right)$$
$$\le \frac{1}{\varepsilon^2 t_n^4}\max_{1\le l\le n} E\left|\sum_{i=1}^{l}[X_i^2 I(|X_i|\le t_n)-E(X_i^2 I(|X_i|\le t_n))]\right|^2$$

and hence by (4.17) we get

$$\frac{1}{t_n^2}\sum_{i=1}^{[nt]} X_i^2 I(|X_i|\le t_n)\xrightarrow{P} t \text{ as } n\to\infty, \text{ uniformly in } t,\ 0\le t\le 1. \tag{4.23}$$

By (4.15)

$$\frac{1}{t_n^2}\sum_{i=1}^{[nt]} X_i^2 I(|X_i|> t_n)\xrightarrow{P} 0 \text{ as } n\to\infty, \text{ uniformly in } t,\ 0\le t\le 1. \tag{4.24}$$

By (4.13), (4.23) and (4.24)

$$\frac{\sum_{i=1}^{[nt]} X_i^2}{\sum_{i=1}^{n} X_i^2}\xrightarrow{P} t \text{ as } n\to\infty, \text{ uniformly in } t,\ 0\le t\le 1,$$

and hence (2.2) follows.

Now let $\varepsilon>0$ be arbitrary but fixed. Then by (4.10) we get

$$P(X_n^*>\varepsilon t_n)\to 0 \text{ as } n\to\infty.$$

Hence (2.3) holds and the corollary follows.

5 Concluding remarks

(1) The condition (2.2) is superfluous in case of non-functional version of Theorem 1.

(2) We have assumed in Corollary 7 that X_1 has a continuous positive spectral density to apply Lemma 1 of Bradley (1992). Our conjecture is that this can be relaxed.

(3) Corollary 7 may be extended for random field as well. Suppose d is a positive integer and $X=(X_n : n\in Z^d)$ is a centered, strictly stationary, non-degenerate random field such that $H(a)=EX_1^2 I(|X_1|\le a)$ is slowly varying as $a\to\infty$, $\rho^*(1)<1$ and $\rho^*(s)\to 0$ as $s\to\infty$, where ρ^* is a "unrestricted mixing" coefficient. Then similar result may be obtained using Bradley (1992).

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