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On hitting and cover times via electric networks

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Abstract: We derive a formula for the sum of expected hitting times along a tour of vertices (i.e., a set of vertices $i_1, i_2, \ldots i_N, i_{N+1} = i_1$) on a graph, that yields as corollaries a number of classical results concerning expected hitting times and a new bound for the expected cover time of the simple random walk on a graph $G = (V, E)$, namely: if R_{ij} is the effective resistance between vertices i and j and R_{tour} is the minimum value of $\sum_{j=1}^{N} R_{i_j, i_{j+1}}$, over all tours that include all vertices in V, then the expected cover time is bounded above by $|E|R_{tour}$.

Key words: Cover times; effective resistance.

1 Introduction

On a connected undirected graph $G = (V, E)$ such that the edge between vertices i and j is given a resistance r_{ij} (or equivalently, a conductance $C_{ij} = 1/r_{ij}$), we can define the random walk on G as the Markov chain $X_n, n \geq 0$, that from its current vertex v jumps to the neighboring vertex w with probability $p_{vw} = C_{vw}/C(v)$, where $C(v) = \sum_{w:w\sim v} C_{vw}$, and $w \sim v$ means that w is a neighbor of v. There may be a conductance C_{zz} from a vertex z to itself, giving rise to a transition probability from z to itself, though the most studied case of these random walks on graphs, the simple random walk, excludes the loops and considers all r_{ij}'s to be equal to 1. Some notation: E_aT_b and E_aC denote the expected value, starting from the vertex a, of respectively, the hitting time T_b of the vertex b and the cover time C, i. e., the number of jumps needed to visit all the vertices in V; R_{ab} is the effective resistance, as computed by means of Ohm's law, between vertices a and b.

The beginner's handbook when studying random walks on graphs from the viewpoint of electric networks is the book of Doyle and Snell (1984). Another mandatory reference in this context is the article of Tetali (1991) where a clever use of the superposition principle and the reciprocity theorem for electric networks yields the following formula for the expected hitting times:

$$E_aT_b = \frac{1}{2}\sum_z C(z)[R_{ab} + R_{bz} - R_{az}] \tag{1.1}$$

where $C(z) = C_{zz} + \sum_{x:z\sim x} C_{zx}$ and where the C_{zz}'s do not intervene in the computation of the effective resistances.

We have used this elegant formula to derive closed form expressions for expected hitting times of the Ehrenfest urn, birth and death chains and circular Markov chains (Palacios, 1994, Palacios, 2000, Palacios and Tetali, 1996), though this list is personal and by no means exhaustive of its possible applications. In this article we use (1.1) to derive a formula for the sum of expected hitting times along a tour of vertices (i. e., a set of vertices $i_1, i_2, \ldots i_N, i_{N+1} = i_1$) that yields as corollaries a number of classical results concerning expected hitting times and a new bound for the expected cover time of a random walk on a graph $G = (V, E)$, namely: if R_{ij} is the effective resistance between vertices i and j and R_{tour} is the minimum value of $\sum_{j=1}^{N} R_{i_j,i_{j+1}}$, over all tours including all vertices in V, then the expected cover time is bounded above by $|E|R_{tour}$.

2 Expected hitting times along a tour

Let $i_1, i_2, \ldots i_N, i_{N+1} = i_1$ be a tour on V (notice that we are not asking that $i_j \sim i_{j+1}$, so that a tour is not necessarily a *cycle*). For $1 \leq j \leq N$, let σ_j be a permutation on the two-element set $\{i_j, i_{j+1}\}$. Then either σ_j is the identity, that we denote ι, or it is the transposition, that we denote τ. Let $sgn(\sigma_j)$ denote the sign of σ_j. Then we have

Theorem 2.1 *For any tour $i_1, i_2, \ldots i_N, i_{N+1} = i_1$, and for any choice of permutations $\sigma_1, \ldots, \sigma_N$:*

$$\sum_{j=1}^{N} sgn(\sigma_j) E_{\sigma_j i_j} T_{\sigma_j i_{j+1}} = \frac{1}{2} \sum_z C(z) \sum_{j=1}^{N} sgn(\sigma_j) R_{\sigma_j i_j, \sigma_j i_{j+1}}. \tag{2.1}$$

Proof. Write every summand in the left hand side of (2.1), using (1.1), as a sum over z of an expression containing three effective resistances. Check that all the effective resistances with a z subscript cancel out in a telescoping fashion, and get the remaining effective resistances out of the z-sum •

Notice that the term $\frac{1}{2}\sum_z C(z)$ in (2.1) reduces to $|E|$ in the case of simple random walk. Now we look at interesting particular cases of (2.1):

(a) $N = 2$, $\sigma_1 = \sigma_2 = \iota$ yields:

$$E_i T_j + E_j T_i = \sum_z C(z) R_{ij}, \tag{2.2}$$

which is a formula for the expected commute time that dates back, at least, to Chandra et al. (1989). Let N_k^{ij} be the number of visits to vertex k for a walk from i to j. In Doyle and Snell (1984) it is proved with elementary means that $N_k^{ij}/C(k)$ is the voltage at k when a unit current flows from i to j. Therefore,

$N_i^{ij}/C(i)$ is the voltage at i when a unit current flows from i to j, which is by definition the effective resistance R_{ij}. Thus we can replace the right hand side of (2.2) with

$$\frac{\sum_z C(z)}{C(i)} N_i^{ij} = \frac{N_i^{ij}}{\pi(i)},$$

where $\pi(i) = \frac{C(i)}{\sum_z C(z)}$ is the stationary distribution of the walk at the vertex i. Thus we have an electric proof of one half of proposition 9-58 of Kemeny et al. (1976) in the context of general random walks on graphs.

(b) $N = 3, \sigma_1 = \sigma_2 = \iota, \sigma_3 = \tau$ leads to:

$$E_i T_j + E_j T_k - E_i T_k = \frac{1}{2} \sum_z C(z)[R_{ij} + R_{jk} - R_{ik}]. \tag{2.3}$$

Now, the triangle inequality for resistances given in Tetali (1991) states that $R_{ij} + R_{jk} - R_{ik} = 2\frac{N_k^{ij}}{C(k)}$, and thus replacing this equation into the right hand side of (2.3) we get

$$E_i T_j + E_j T_k - E_i T_k = \frac{N_k^{ij}}{\pi(k)},$$

which is the second half of the proposition 9-58 mentioned above. A similar equation in Chung (1967) (theorem 3, p. 65) can be proved along the same lines.

(c) Take all $\sigma_j = \iota$. The sum of the expected times over a tour is the same, regardless of whether the traversal is "clockwise" or "counterclockwise", because of the form of the right hand side of (2.1) and the fact that $R_{ij} = R_{ji}$. This says, for instance in the case of $N = 3$ that

$$E_i T_j + E_j T_k + E_k T_i = E_k T_j + E_j T_i + E_i T_k,$$

a result well known in the literature (see for instance, Coppersmith et al., 1993).

(d) Take $i_1, \ldots, i_{2|V|-1} = i_1$ to be a cyclic traversal of the vertices in V along a spanning tree. Take all $\sigma_j = \iota$, and for simplicity, let us consider the simple random walk case. We get

$$\sum_{j=1}^{2|V|-2} E_{i_j} T_{i_{j+1}} = 2|E| \sum_{j=1}^{2|V|-2} R_{i_j, i_{j+1}}. \tag{2.4}$$

The left hand side of (2.4) is evidently an upper bound for the expected cover-and-return time. When the summation in the right hand side is minimized over all spanning trees, one gets the quantity R_{span} and the upper bound $2|E|R_{span}$ for the expected cover time that was first introduced in Chandra et al. (1989),

and subsequently used by Feige (1995) and Coppersmith et al. (1996) to obtain maximal results for expected cover times. Notice that the factor 2 in (2.4) comes from the fact that every edge of the spanning tree is traversed twice.

(e) As in (d), consider the simple random walk case and all $\sigma_j = \iota$. Take $i_1, \dots i_N, i_{N+1} = i_1$ to be any tour that contains all vertices in V. Then we get

$$\sum_{j=1}^{N} E_{i_j} T_{i_{j+1}} = |E| \sum_{j=1}^{N} R_{i_j, i_{j+1}}. \tag{2.5}$$

The left hand side of (2.5) is, again evidently, an upper bound for the expected cover-and-return time. When the summation in the right hand side is minimized over all tours that contain all vertices in V, one gets the constant R_{tour}, and the upper bound $|E|R_{tour}$ for $E_v C$ that improves the bound $2|E|R_{span}$, since

$$R_{tour} \leq 2R_{span}, \tag{2.6}$$

because a cyclic traversal of a spanning tree is just a special kind of tour.

It is worth mentioning that whereas applications (a) and (d) of theorem 2.1 presented here rely only on the electrical expression for the commute time, for which only the superposition principle is needed, the other applications require the full strength of Tetali's formula (1.1).

Upper bounds for expected cover times based on spanning trees date back to the seminal paper of Aleliunas et al. (1979), where they gave the bound $E_v C \leq 2|E|(|V|-1)$. It is obvious that

$$|E|R_{tour} \leq 2|E|R_{span} \leq 2|E|(|V|-1),$$

and that if the graph G is a tree both inequalities are equalities and the common value of the bounds is $2(|V|-1)^2$. Otherwise we can prove the following

Proposition 2.1 *If G is not a tree, then*

$$|E|R_{tour} < 2|E|(|V|-1).$$

Proof. G contains at least one k-cycle B, $k \geq 3$. The subgraph $G - B$ may be composed of several connected components, but for simplicity we will assume that there is only one. Since G is connected, there are $v \in B$ $w \in G - B$ such that $(v, w) \in E$. Let Q be a spanning tree of the subgraph $G - B$. The tour composed of a traversal of B started and ended at v, followed by a traversal of (v, w), followed by a cyclic traversal of Q started and ended at w, followed by a traversal of (w, v) satisfies $R_{tour} \leq k + 2 + 2(|V| - k - 1) = 2|V| - k < 2(|V| - 1)$. The general case is treated similarly •

A very basic example where the inequality in (2.6) is strict, and thus where the new bound $|E|R_{tour}$ improves $2|E|R_{span}$, is the n-cycle graph, for which $2|E|R_{span} = 2(n-1)^2$, and $|E|R_{tour} = n(n-1)$. In fact, we can extend these ideas to prove the following

Proposition 2.2 *If $G = (V, E)$ is edge-transitive, then*

$$E_v C \leq L(|V| - 1),$$

where L is the length of the shortest tour containing all vertices, satisfying $|V| \leq L \leq 2(|V| - 1)$.

Proof. The fact that G is edge-transitive means that for each pair of (unoriented) edges there is a graph automorphism sending one edge onto the other. This implies immediately that the effective resistance of all edges is the same; this fact plus the well known Foster's theorem stating that

$$\sum_{(a,b)\in E} R_{ab} = |V| - 1,$$

imply that for every edge $(a, b) \in E$:

$$R_{ab} = \frac{|V| - 1}{|E|}.$$

Therefore, $|E|R_{tour} = |E|L\left(\frac{|V| - 1}{|E|}\right)$, ending the proof •

The above proposition improves the bound for the expected cover time of edge-transitive graphs given in Palacios and Renom (1998). The upper bound for L is attained for trees and the lower bound for Hamiltonian graphs; in particular, if G is edge-transitive and regular, $E_v C < 2|V|(|V| - 1)$. Note that Hamiltonicity by itself does not guarantee low cover times: if we modify the well known lollipop graph so that the free end of its linear part is attached to its complete part, we obtain a Hamiltonian graph with cubic hitting (and thus cover) times. (See Brightwell and Winkler (1990) for related details). Along these lines, we can delete the 2 factor in the classic bound $2|E|(|V| - 1)$, in the case of Hamiltonian graphs:

Proposition 2.3 *If $G = (V, E)$ is Hamiltonian, then*

$$E_v C \leq |E|(|V| - 1).$$

Proof. Delete from G all edges except those in the Hamiltonian cycle, in order to obtain a new (cycle) graph B. By the monotonicity principle (see Doyle and Snell, 1984), the original effective resistance of an edge in the cycle in G is bounded by the effective resistance of that edge in B, and that is easily computed as $\frac{|V|-1}{|V|}$. So R_{tour} is bounded by $|V| - 1$, ending the proof •

Proposition 2.1 suggests that the more cycles a graph has, the better the bound $|E|R_{tour}$ is over the classical bounds. A simple example, not covered by the above propositions, showing how $|E|R_{tour}$ at least shaves a factor of 2 off the bound $2|E|R_{span}$, can be constructed replacing each edge of an m-edge linear

graph with a k-cycle. For this graph we get $2|E|R_{span} = 2m^2(k-1)^2$, whereas $|E|R_{tour} = m^2k(k-1)$.

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