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Testing of unit and fractional roots in the context of deterministic trends with weakly autocorrelated disturbances

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Abstract: In this article we propose the use of a modified version of a procedure suggested by Robinson (1994) for testing $I(d)$ statistical models in the presence of deterministic regressors and weakly autocorrelated disturbances. The new test has better finite-sample properties than Robinson's (1994) original test. An empirical application, showing the performance of the test in the context of deterministic trends is also included at the end of the article.

Key words: Deterministic trends, fractional integration, long memory.

1 Introduction

Most of the commonly-used unit-root tests existing in the literature, (e.g., Dickey and Fuller, 1979, Phillips and Perron, 1988, the alternative in Kwiatkowski et al., 1992, etc.) have been developed in autoregressive (AR) models of form:

$$(1 - \rho L)x_t = u_t, \quad t = 1, 2, \ldots, \tag{1.1}$$

where L is the lag-operator (i.e. $Lx_t = x_{t-1}$) and u_t is an $I(0)$ process, defined as a covariance stationary process with spectral density function that is positive and finite at the zero frequency. Then, the unit root null corresponds to:

$$H_0 : \rho = 1. \tag{1.2}$$

Conspicuous features of these methods for testing unit roots are the non-standard nature of the null asymptotic distributions which are involved, and the absence of Pitman efficiency theory. However, these properties are not automatic, rather depending on what might be called a degree of "smoothness" in the model across the parameters of interest, in the sense that the limit distribution does not change in an abrupt way with small changes in the parameters. This is associated with the radically variable long run properties of AR processes around the unit root. Under (1.1), for $|\rho| > 1$, x_t is explosive, for $|\rho| < 1$, x_t is covariance stationary, and for $\rho = 1$ it is nonstationary but non-explosive. In view of these abrupt changes, the fractional processes have become a rival alternative to the AR model in case of testing for unit roots. Robinson (1994) developed a test for unit roots, that

unlike these previous procedures, was not embedded in AR structures, but based on fractional alternatives of form:

$$(1-L)^d x_t = u_t, \quad t = 1, 2, \ldots, \tag{1.3}$$

where d can be any real number, u_t is $I(0)$, and where the unit root null corresponds to the null:

$$H_0 : d = 1. \tag{1.4}$$

Fractional and AR departures from (1.1) and (1.3) have very different long-run implications. In (1.3), x_t is nonstationary but non-explosive for all $d \geq 0.5$. As d increases beyond 0.5 and through 1, x_t can be viewed as becoming "more nonstationary" (in the sense, for example, that the variance of partial sums increases in magnitude), but it does so gradually, unlike in case of (1.1) around (1.2).

Both, the AR and the fractional methods of testing unit (or fractional) roots allow us to incorporate deterministic components such as an intercept or a linear time trend and thus, deterministic trends can be included in a regression model. However, while the limit distribution in those procedures based on AR alternatives changes with the features of these regressors, (see also, Schmidt and Phillips, 1992), it remains standard with Robinson's (1994) procedure. This is important because many observed time series exhibit linear trends. Examples include GNP, consumption, price level or global temperatures. For such series, differenced data often appear stationary and exhibit long memory and a non-zero mean (see, e.g. Hassler and Wolters, 1995, and Deo and Hurvich, 1998), implying a linear trend in the original data. For an overview about the problem of distinguishing long memory and deterministic trends see Sibbertsen and Venetis (2003), and another recent empirical investigation is Hassler, Marmol and Velasco (2003).

In this article, we show how the test presented in Robinson (1994) can be improved in the context of weakly autocorrelated disturbances. We show that estimating the regression coefficients by generalized least squares (GLS) rather than ordinary least squares (OLS), the test presents better statistical properties in finite samples. The outline of the paper is as follows: Section 2 briefly describes the test of Robinson (1994) and we also present the new test statistic based on GLS estimation. Monte Carlo experiments are presented in Section 3 to show the performance of the new test in finite samples relative to Robinson (1994). Section 4 contains a simple empirical application and Section 5 concludes.

2 Testing of unit and fractional roots

Following discussions of Bhargava (1986), Schmidt and Phillips (1992) and others of parametrization of unit-root models, we consider the model:

$$y_t = \beta' z_t + x_t \ , \quad t = 1, 2, \ldots, \tag{2.1}$$

where y_t is the observed time series; β is a $(k \times 1)$ vector of unknown parameters; z_t is a $(k \times 1)$ vector of deterministic regressors, that may include, for example,

an intercept ($z_t \equiv 1$) or an intercept and a linear time trend ($z_t = (1, t)'$), and x_t is given by (1.3).

Robinson (1994) proposed a Lagrange Multiplier (LM) test, where the null hypothesis is:

$$H_0 : d = d_0 \ , \tag{2.2}$$

in a model given by (1.3) and (2.1) for any given real value d_0. To be slightly general, we suppose that u_t in (1.3) follows an autocorrelated structure of form:

$$\varphi(L)u_t = \varepsilon_t \ , \quad t = 1, 2, \ldots, \tag{2.3}$$

where $\varphi(L)$ may be a stationary AR polynomial, the MA representation of an invertible process or a combination of both, with white noise ε_t. Thus, using (1.3), (2.1) and (2.3), the set-up in Robinson (1994) becomes:

$$y_t = \beta' z_t + x_t \ ; \quad (1-L)^d x_t = u_t \ ; \quad \varphi(L)u_t = \varepsilon_t \ , \tag{2.4}$$

which may be rewritten as:

$$(1-L)^d y_t = \beta'(1-L)^d z_t + u_t \ ; \quad \varphi(L)u_t = \varepsilon_t \ . \tag{2.5}$$

The test statistic most used in the empirical work (see, e.g., Gil-Alana and Robinson, 1997; Gil-Alana, 2000, 2001a; etc.) is the one based on the OLS residuals:

$$\hat{u}_t = (1-L)^{d_0} y_t - \hat{\beta}' w_t \ , \quad t = 1, 2, \ldots, \tag{2.6}$$

where

$$\hat{\beta} = \left(\sum_{t=1}^{T} w_t w_t' \right)^{-1} \sum_{t=1}^{T} w_t (1-L)^{d_0} y_t \ ; \quad w_t = (1-L)^{d_0} z_t \ . \tag{2.7}$$

Specifically, the test statistic is given by:

$$\hat{r} = \left(\frac{T}{\hat{A}} \right)^{1/2} \frac{\hat{a}}{\hat{\sigma}^2} \ , \tag{2.8}$$

where T is the sample size and

$$\hat{a} = -\frac{2\pi}{T} \sum_{j=1}^{T-1} \psi(\lambda_j) g(\lambda_j; \hat{\tau})^{-1} I(\lambda_j) \ ; \quad \hat{\sigma}^2 = \sigma^2(\hat{\tau}) = \frac{2\pi}{T} \sum_{j=1}^{T-1} g(\lambda_j; \hat{\tau})^{-1} I(\lambda_j) \ ;$$

$$\hat{A} = \frac{2}{T} \left(\sum_{j=1}^{T-1} \psi(\lambda_j)^2 - \sum_{j=1}^{T-1} \psi(\lambda_j) \hat{\varepsilon}(\lambda_j)' \left(\sum_{j=1}^{T-1} \hat{\varepsilon}(\lambda_j) \hat{\varepsilon}(\lambda_j)' \right)^{-1} \sum_{j=1}^{T-1} \hat{\varepsilon}(\lambda_j) \psi(\lambda_j) \right) ;$$

$$\psi(\lambda_j) = \log \left| 2 \sin \frac{\lambda_j}{2} \right| \ ; \ \hat{\varepsilon}(\lambda_j) = \frac{\partial}{\partial \tau} \log g(\lambda_j; \hat{\tau}) \ ; \ \lambda_j = \frac{2\pi j}{T}; \ \hat{\tau} = \arg\min_{\tau \in H} \sigma^2(\tau) \ ,$$

where H is a compact subset of the R^q Euclidean space. The function g above is a known function coming from the spectral density function of u_t,

$$f(\lambda, \sigma^2, \tau) = \frac{\sigma^2}{2\pi} g(\lambda, \tau) \ , \quad -\pi < \lambda \leq \pi \ ,$$

and $I(\lambda_j)$ is the periodogram of $\hat{u}_t$. Note that these tests are purely parametric and therefore, they require specific modelling assumptions to be made regarding the short memory specification of u_t. Thus, if u_t is white noise, $g \equiv 1$, $(\hat{\varepsilon}(\lambda_j) = 0)$, and if u_t is an AR process of form $\varphi(L)u_t = \varepsilon_t$, then $g = |\varphi(e^{i\lambda})|^{-2}$, with $\sigma^2 = V(\varepsilon_t)$, so that the AR coefficients are a function of τ.

Based on H_0 (2.2), Robinson (1994) showed that under certain regularity conditions[1]

$$\hat{r} \rightarrow_d N(0,1) \ \text{ as } \ T \rightarrow \infty \ . \tag{2.9}$$

Thus, an approximate $100\alpha\%$ level test of (2.2) will reject H_0 against the alternative: $H_a : d > d_0$ $(d < d_0)$ if $\hat{r} > z_\alpha$ $(\hat{r} < -z_\alpha)$, where the probability that a standard normal variate exceeds z_α is α. He also showed that the test is efficient in the Pitman sense, i.e., that against local alternatives of form: $H_a : d = d_0 + \delta T^{-1/2}$, with $\delta \neq 0$, $\hat{r}$ has a limit normal distribution with variance 1 and mean that cannot (when u_t is Gaussian) be exceeded in absolute value by that of any rival regular statistic. This version of the test of Robinson (1994) looks at unit (and fractional) roots at the long run or zero frequency. There exist other versions of his test based on seasonal (e.g., Gil-Alana, 1999, Gil-Alana and Robinson, 2001) and cyclical (Gil-Alana, 2001b) roots. However, given the importance of the trending or long-run component in macroeconomic series, we have decided to concentrate in this article only on processes with the pole in the spectrum occurring exclusively at the long run or zero frequency.

We describe the new version of the test. Based on the same set-up as in Robinson (1994), we can rewrite (2.4) as:

$$\varphi(L)(1-L)^d y_t = \beta' \psi(L)(1-L)^d z_t + \varepsilon_t \ . \tag{2.10}$$

and compute, under H_0 (2.2), the least squares estimate of β by:

$$\tilde{\beta} = \left(\sum_{t=1}^{T} w_t^* w_t^{*'} \right)^{-1} \sum_{t=1}^{T} w_t^* \hat{\varphi}(L)(1-L)^{d_0} y_t \ ; \quad w_t^* = \hat{\varphi}(L)(1-L)^{d_0} z_t \ , \tag{2.11}$$

where $\hat{\varphi}$ is the autocorrelated polynomial once it has been estimated based on $\hat{u}_t$. The new test statistic will have the same functional form as the one given by (2.8) but based on the new residuals:

$$\tilde{u}_t = (1-L)^{d_0} y_t - \tilde{\beta}' w_t \ , \quad t = 1, 2, \ldots, \tag{2.12}$$

and the test statistic will still remain with the same standard null and local limit distributions by reasons described in Robinson (1994). In the following section,

[1] These conditions are very mild regarding technical assumptions which are satisfied by model (1.3) and (2.1).

a very simple Monte Carlo experiment is conducted to examine the finite sample properties of the new test relative to Robinson (1994). A diskette containing the Fortran codes for the programmes is available from the author upon request.

3 A Monte Carlo simulation study

We will use R1 to denominate the statistic given by $\hat{r}$ in (2.8), based on the residuals $\hat{u}_t$ in (2.6), i.e., corresponding to the test statistic suggested by Robinson (1994) and used in most of the empirical works, while R2 refers to the new statistic based on $\tilde{u}_t$ in (2.12).

Across Tables 1-3 we consider the model:

$$y_t = 1 + t + x_t \ ; \ \ (1-L)^d x_t = u_t \ ; \ \ u_t = \phi u_{t-1} + \varepsilon_t \ , \tag{3.1}$$

with values $d = 0, (0.50), 2$; $\phi = -0.75, -0.25, 0.25$ and 0.75; white noise ε_t, and $T = 70, 300$ and 1000. We test $H_0 : d = d_0$ in a model given by:

$$y_t = \alpha + \beta t + x_t \ ; \ \ (1-L)^d x_t = u_t \ ; \ \ u_t \sim \mathrm{AR}(1) \tag{3.2}$$

and d_0 equal in all cases to the true value. We look at the sizes of the tests when directed against the one-sided alternatives $H_a : d < d_0$ and $H_a : d > d_0$. In addition, we also look at the mean and the variance of the regression coefficients in (3.2). We generate Gaussian series using the routines GASDEV and RAN3 of Press, Flannery, Teukolsky and Wetterling (1986), the nominal size is 5% and 10,000 replications are used in each case.

Starting with $T = 70$, (in Table 1), we see that using the new version of the test (R2), the values are still below the nominal value, though they are closer to the 5% level than when using R1. If we look at the regression coefficients, we observe that they are also closer to 1 when R2 is employed. The same experiment was also conducted with $T = 100$ and the results were very similar to the previous case, observing only a slight improvement with respect to the values in Table 1.

Table 2 reports the results with $T = 300$. We observe that the sizes improve in most of the cases, approximating to the nominal value. Similarly to Table 1, R2 outperforms R1 in a number of cases in relation to the size. The mean and variance of the regression coefficients also improve when the new tests are employed, though the proportion of cases with an improvement seems to be smaller than in Table 1, implying that as we increase the sample size, the two methods have similar behaviour. This is corroborated by the results in Table 3, which correspond to the case with $T = 1000$. Here, we observe few cases with an improvement, and both methods produce satisfactory results in terms of size and mean and variance of the regression coefficients.

Tables 4 and 5 report the rejection frequencies of the tests in the context of fractional alternatives. In Table 4, we assume that the true model is given by the model in (3.1) with $d = 1$, (i.e. containing a unit root), and again $\phi = -0.75, -0.25, 0.25$ and 0.75. We look at alternatives of form as (3.2), testing H_0 (2.2) with $d_0 = 0, 0.25, 0.50$ and 0.75 (in which case, the appropriate alternative

Table 1 *Empirical sizes and mean and variance of the regression coefficients using R1, R2 and $T = 70$. Model: $y_t = 1 + t + x_t$; $(1-L)^d x_t = u_t$; $u_t = \phi u_{t-1} + \varepsilon_t$.*

ϕ	d_0	Test statistic	Alternatives		Mean		Variance	
			$H_a: d > d_0$	$H_a: d < d_0$	α	β	α	β
-0.75	0.00	R1	0.010	0.079	0.9927	0.9989	0.0155	0.0006
		R2	**0.044**	**0.072**	**0.9928**	**1.0000**	**0.0143**	**0.0005**
	0.50	R1	0.012	0.060	0.9712	0.9994	0.2655	0.0009
		R2	**0.045**	**0.047**	**0.9834**	**0.9998**	**0.2121**	**0.0007**
	1.00	R1	0.013	0.041	0.9744	0.9960	1.0110	0.0059
		R2	**0.016**	**0.050**	**0.9819**	0.9960	**0.8796**	**0.0038**
	1.50	R1	0.014	0.047	0.9800	0.9779	1.1127	0.1081
		R2	0.014	**0.051**	**0.9904**	**0.9791**	**0.8004**	**0.0993**
	2.00	R1	0.015	0.044	0.9711	0.9911	1.0445	1.0229
		R2	**0.016**	**0.046**	**0.9909**	0.9910	**0.8708**	**0.8124**
-0.25	0.00	R1	0.008	0.057	0.9884	0.9984	0.0298	0.0011
		R2	**0.009**	0.057	**0.9900**	**0.9996**	**0.0267**	**0.0009**
	0.50	R1	0.009	0.031	0.9704	0.9994	0.4456	0.0008
		R2	**0.011**	**0.042**	**0.9711**	0.9994	**0.0789**	**0.0003**
	1.00	R1	0.012	0.025	0.9745	0.9933	1.0241	0.0088
		R2	0.012	**0.028**	**0.9756**	**0.9937**	**0.9959**	**0.0074**
	1.50	R1	0.009	0.030	0.9905	0.9656	1.0118	0.3234
		R2	**0.010**	**0.032**	**0.9933**	**0.9660**	**1.0359**	**0.3134**
	2.00	R1	0.013	0.028	0.9804	0.9759	1.1290	1.4855
		R2	**0.015**	**0.029**	**0.9906**	0.9751	**1.0456**	**1.4702**
0.25	0.00	R1	0.009	0.018	0.9800	1.0004	0.0744	0.0003
		R2	**0.011**	**0.023**	**0.9812**	1.0004	**0.0654**	**0.0002**
	0.50	R1	0.009	0.020	0.9597	0.9991	0.8095	0.0004
		R2	**0.010**	0.020	**0.9609**	**0.9992**	**0.7987**	0.0004
	1.00	R1	0.010	0.023	0.9765	0.9895	1.1209	0.0199
		R2	**0.011**	**0.024**	**0.9809**	**0.9909**	**1.0651**	**0.0196**
	1.50	R1	0.010	0.021	1.0324	0.9411	0.9004	0.7906
		R2	**0.011**	**0.023**	**1.0221**	**0.9417**	**0.8975**	**0.7789**
	2.00	R1	0.011	0.024	1.0109	0.9644	1.0055	1.1172
		R2	0.011	0.024	**1.0103**	**0.9795**	**0.9807**	**1.0927**
0.75	0.00	R1	0.007	0.015	0.9443	1.0003	0.3211	0.0004
		R2	0.007	**0.017**	**0.9517**	**1.0003**	**0.2611**	**0.0003**
	0.50	R1	0.008	0.015	0.9118	0.9954	1.1130	0.0033
		R2	**0.011**	**0.018**	**0.9311**	**0.9966**	**1.0788**	0.0033
	1.00	R1	0.008	0.020	0.9901	0.9657	1.2391	0.2343
		R2	**0.011**	0.020	**0.9932**	**0.9709**	**0.9088**	**0.1908**
	1.50	R1	0.011	0.020	1.0904	0.8132	0.9908	1.0042
		R2	**0.014**	**0.022**	**1.0765**	**0.8653**	**0.9890**	**0.9076**
	2.00	R1	0.011	0.019	1.0431	0.9486	1.0603	1.0499
		R2	**0.012**	**0.020**	**1.0223**	**0.9599**	**0.8054**	**0.9313**

Those cases where R2 is superior to R1 appear in bold type.

Table 2 *Empirical sizes and mean and variance of the regression coefficients using R1, R2 and $T = 300$. Model: $y_t = 1 + t + x_t$; $(1-L)^d x_t = u_t$; $u_t = \phi u_{t-1} + \varepsilon_t$.*

ϕ	d_0	Test statistic	Alternatives		Mean		Variance	
			$H_a: d > d_0$	$H_a: d < d_0$	α	β	α	β
-0.75	0.00	R1	0.033	0.071	0.9969	1.0000	0.0047	0.00002
		R2	0.033	0.071	**0.9971**	1.0000	**0.0046**	**0.00001**
	0.50	R1	0.034	0.064	0.9768	1.0000	0.2140	0.000003
		R2	0.039	**0.062**	**0.9830**	0.9999	**0.1751**	0.000003
	1.00	R1	0.038	0.052	0.9596	0.9986	0.7765	0.0010
		R2	0.035	0.058	**0.9818**	0.9986	**0.5283**	0.0010
	1.50	R1	0.033	0.059	0.9602	0.9860	0.5348	0.1410
		R2	**0.034**	**0.055**	**0.9849**	0.9830	**0.4309**	**0.1375**
	2.00	R1	0.033	0.055	0.9548	1.0034	0.5116	1.0157
		R2	0.033	0.058	**0.9791**	0.9906	**0.4328**	**0.8408**
-0.25	0.00	R1	0.032	0.059	0.9957	1.0000	0.0091	0.000001
		R2	0.032	0.061	**0.9959**	1.0000	0.0091	0.000001
	0.50	R1	0.033	0.059	0.9707	0.9999	0.3408	0.0000002
		R2	0.031	0.061	**0.9729**	0.9999	**0.3281**	0.000001
	1.00	R1	0.032	0.043	0.9602	0.9980	0.5457	0.0020
		R2	**0.039**	**0.046**	**0.9657**	0.9980	**0.3312**	0.0020
	1.50	R1	0.035	0.050	0.9715	0.9767	0.4505	0.2673
		R2	0.035	0.050	**0.9751**	0.9756	**0.3382**	**0.2669**
	2.00	R1	0.032	0.046	0.9757	0.9825	0.5790	0.7580
		R2	0.031	**0.047**	**0.9775**	**0.9899**	**0.4888**	**0.6263**
0.25	0.00	R1	0.035	0.041	0.9931	1.0000	0.0253	0.0000003
		R2	0.035	**0.043**	**0.9998**	1.0000	**0.0252**	**0.000007**
	0.50	R1	0.037	0.038	0.9579	0.9999	0.7115	0.000002
		R2	**0.038**	**0.044**	0.9532	0.9999	**0.6706**	0.000002
	1.00	R1	0.033	0.041	0.9616	0.9966	0.6542	0.0057
		R2	0.033	**0.045**	**0.9892**	**0.9967**	**0.5586**	0.0057
	1.50	R1	0.039	0.041	0.9893	0.9577	0.6692	0.6875
		R2	**0.031**	**0.042**	**0.9971**	0.9566	**0.5700**	**0.6779**
	2.00	R1	0.035	0.038	0.9965	0.9617	0.4677	0.5549
		R2	0.035	0.035	**0.9948**	**0.9639**	**0.2573**	**0.4868**
0.75	0.00	R1	0.031	0.039	0.9796	1.0000	0.2186	0.0000003
		R2	0.031	**0.043**	0.9741	1.0000	**0.1975**	**0.0000001**
	0.50	R1	0.030	0.038	0.9041	0.9997	3.6760	0.0002
		R2	**0.038**	**0.040**	**0.9486**	0.9996	**0.9546**	0.0002
	1.00	R1	0.031	0.045	0.9684	0.9898	1.0175	0.0516
		R2	0.031	**0.051**	**0.9724**	0.9895	**0.6437**	**0.0507**
	1.50	R1	0.039	0.036	1.0407	0.8814	0.8820	0.4509
		R2	0.038	0.035	**1.0102**	**0.9331**	**0.6913**	**0.3121**
	2.00	R1	0.035	0.034	1.0174	0.9408	0.6334	0.5664
		R2	0.032	**0.043**	**1.0044**	**0.9604**	**0.5111**	**0.2309**

Those cases where R2 is superior to R1 appear in bold type.

Table 3 *Empirical sizes and mean and variance of the regression coefficients using R1, R2 and T = 1000. Model:* $y_t = 1 + t + x_t$; $(1-L)^d x_t = u_t$; $u_t = \phi u_{t-1} + \varepsilon_t$.

ϕ	d_0	Test statistic	Alternatives		Mean		Variance	
			$H_a: d > d_0$	$H_a: d < d_0$	α	β	α	β
-0.75	0.00	R1	0.042	0.054	0.9993	1.0000	0.0003	0.000002
		R2	0.042	0.054	**0.9995**	1.0000	0.0003	0.00002
	0.50	R1	0.045	0.053	0.9890	1.0000	0.0014	0.00004
		R2	**0.046**	**0.052**	0.9800	0.9999	0.0017	0.00004
	1.00	R1	0.044	0.052	0.9596	0.9999	0.0413	0.0008
		R2	0.044	0.052	**0.9904**	0.9999	**0.0234**	0.0009
	1.50	R1	0.046	0.053	0.9902	0.9894	0.1534	0.0943
		R2	0.046	0.053	**0.9957**	**0.9917**	0.1659	0.1098
	2.00	R1	0.045	0.051	0.9890	1.0005	0.1332	0.5101
		R2	**0.046**	0.052	**0.9899**	0.9996	0.1332	**0.3547**
-0.25	0.00	R1	0.041	0.053	0.9995	1.0000	0.0009	0.000001
		R2	0.041	0.053	0.9995	1.0000	0.0009	0.000001
	0.50	R1	0.042	0.059	0.9903	1.0000	0.0023	0.000002
		R2	0.042	0.060	**0.9909**	1.0000	**0.0022**	0.000002
	1.00	R1	0.046	0.048	0.9893	0.9998	0.0351	0.00001
		R2	0.046	0.048	0.9893	**0.9999**	0.0364	0.00001
	1.50	R1	0.045	0.050	0.9809	0.9998	0.0324	0.0198
		R2	0.045	0.050	**0.9893**	0.9998	0.0327	**0.0091**
	2.00	R1	0.042	0.049	0.9904	0.9987	0.0241	0.1324
		R2	0.041	0.049	0.9904	**0.9989**	**0.0744**	**0.0298**
0.25	0.00	R1	0.044	0.049	0.9990	1.0000	0.0003	0.000001
		R2	0.044	0.049	**0.9993**	1.0000	0.0003	0.000001
	0.50	R1	0.047	0.048	0.9900	1.0000	0.0015	0.000001
		R2	**0.048**	0.048	0.9900	1.0000	**0.0009**	0.00002
	1.00	R1	0.046	0.044	0.9807	1.0000	0.0321	0.00007
		R2	0.046	**0.045**	0.9802	1.0000	0.0411	0.0008
	1.50	R1	0.049	0.046	0.9901	0.9904	0.0092	0.0175
		R2	0.049	0.046	0.9900	**0.9975**	**0.0071**	**0.0023**
	2.00	R1	0.047	0.048	0.9935	0.9974	0.0151	0.1124
		R2	0.047	0.048	**0.9938**	0.9907	0.0513	0.1233
0.75	0.00	R1	0.042	0.049	0.9903	1.0000	0.0005	0.000002
		R2	0.042	0.049	0.9901	1.0000	0.0005	0.000002
	0.50	R1	0.040	0.048	0.9904	1.0000	0.0006	0.00001
		R2	**0.043**	0.048	**0.9905**	1.0000	0.0006	0.00002
	1.00	R1	0.043	0.045	0.9809	0.9913	0.0873	0.0034
		R2	0.043	**0.049**	0.9779	0.9913	**0.0439**	**0.0013**
	1.50	R1	0.048	0.046	1.0001	0.9123	0.0913	0.0320
		R2	0.048	0.046	1.0004	**0.9267**	**0.0913**	0.1156
	2.00	R1	0.045	0.044	1.0017	0.9678	0.0543	0.0325
		R2	0.045	0.044	**1.0014**	**0.9899**	**0.0499**	0.0934

Those cases where R2 is superior to R1 appear in bold type.

Table 4 *Rejection frequencies of R1 and R2 against fractional alternatives. True model:* $y_t = 1 + t + x_t$; $(1-L)x_t = u_t$; $u_t = \phi u_{t-1} + \varepsilon_t$. *Alternatives:* $y_t = \alpha + \beta t + x_t$; $(1-L)^d x_t = u_t$; $u_t \sim= \mathrm{AR}(1)$; $H_0 : d = d_0$.

$T = 70$		$H_a : d > d_0$				$H_a : d < d_0$			
ϕ	Stat./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
-0.75	R1	0.194	0.716	0.544	0.345	0.566	0.809	0.845	0.917
	R2	**0.205**	**0.751**	**0.687**	**0.432**	**0.623**	**0.811**	**0.899**	**0.923**
-0.25	R1	0.003	0.004	0.113	0.094	0.112	0.265	0.303	0.908
	R2	**0.074**	**0.033**	**0.199**	**0.119**	**0.324**	**0.777**	**0.811**	**0.917**
0.25	R1	0.177	0.013	0.001	0.044	0.097	0.321	0.576	0.817
	R2	**0.192**	**0.046**	**0.019**	**0.076**	**0.190**	**0.412**	**0.702**	**0.900**
0.75	R1	0.306	0.245	0.363	0.055	0.097	0.0004	0.052	0.230
	R2	**0.661**	**0.577**	**0.441**	**0.099**	**0.095**	**0.113**	**0.483**	**0.560**
$T = 100$		$H_a : d > d_0$				$H_a : d < d_0$			
ϕ	Stat./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
-0.75	R1	0.274	0.942	0.956	0.550	0.703	0.996	0.999	0.999
	R2	**0.280**	**0.951**	**0.957**	0.523	**0.723**	0.991	**1.000**	**1.000**
-0.25	R1	0.010	0.007	0.182	0.174	0.482	0.958	0.999	1.000
	R2	**0.114**	**0.047**	**0.231**	**0.179**	**0.500**	**0.962**	0.999	1.000
0.25	R1	0.491	0.027	0.000	0.091	0.132	0.625	0.944	0.995
	R2	**0.745**	**0.246**	**0.069**	**0.095**	**0.133**	**0.634**	**0.945**	0.995
0.75	R1	0.856	0.745	0.560	0.079	0.102	0.003	0.086	0.446
	R2	**0.923**	**0.978**	**0.834**	**0.131**	**0.102**	**0.003**	0.083	**0.455**
$T = 300$		$H_a : d > d_0$				$H_a : d < d_0$			
ϕ	Stat./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
-0.75	R1	0.125	0.963	0.904	0.991	0.998	1.000	1.000	1.000
	R2	**0.227**	**0.977**	**0.968**	0.991	0.998	1.000	1.000	1.000
-0.25	R1	0.041	0.012	0.930	0.856	0.977	1.000	1.000	1.000
	R2	**0.208**	**0.159**	**0.953**	0.856	**0.978**	1.000	1.000	1.000
0.25	R1	0.848	0.044	0.013	0.124	0.734	0.999	1.000	1.000
	R2	**0.978**	**0.313**	**0.152**	**0.165**	0.733	0.999	1.000	1.000
0.75	R1	0.963	0.848	0.833	0.354	0.125	0.171	0.768	0.995
	R2	0.907	**0.998**	**0.996**	**0.435**	0.122	0.162	0.767	0.995
$T = 1000$		$H_a : d > d_0$				$H_a : d < d_0$			
ϕ	Stat./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
-0.75	R1	0.425	0.999	0.999	1.000	1.000	1.000	1.000	1.000
	R2	**0.433**	0.999	0.999	1.000	1.000	1.000	1.000	1.000
-0.25	R1	0.541	0.999	0.995	0.999	1.000	1.000	1.000	1.000
	R2	**0.708**	0.999	**0.999**	0.999	1.000	1.000	1.000	1.000
0.25	R1	0.912	0.422	0.413	0.756	0.909	1.000	1.000	1.000
	R2	**0.999**	0.413	**0.467**	**0.761**	0.909	1.000	1.000	1.000
0.75	R1	0.999	0.999	0.999	0.414	0.809	0.996	0.999	1.000
	R2	0.999	0.999	0.999	**0.768**	**0.813**	0.996	0.999	1.000

Those values where the rejection frequencies corresponding to R2 are superior to those of R1 appear in bold type

Table 5 *Rejection frequencies of R1 and R2 against fractional alternatives. True model:* $y_t = 1 + t + x_t$; $(1-L)x_t = u_t$; $u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \varepsilon_t$. *Alternatives:* $y_t = \alpha + \beta t + x_t$; $(1-L)^d x_t = u_t$; $u_t \sim= \mathrm{AR}(2)$; $H_0 : d = d_0$.

$T = 70$		$H_a : d > d_0$				$H_a : d < d_0$			
(ϕ_1, ϕ_2)	St./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
(0.25, 0.25)	R1	0.009	0.099	0.057	0.011	0.075	0.112	0.203	0.410
	R2	**0.081**	**0.112**	**0.111**	**0.037**	**0.104**	**0.120**	**0.211**	**0.497**
(0.25, -0.25)	R1	0.088	0.111	0.090	0.056	0.051	0.197	0.436	0.778
	R2	**0.101**	**0.123**	**0.094**	**0.039**	**0.058**	**0.211**	0.600	**0.801**
(-0.25, 0.25)	R1	0.002	0.099	0.100	0.039	0.198	0.533	0.807	0.900
	R2	**0.099**	**0.110**	**0.122**	**0.055**	**0.200**	**0.657**	**0.905**	**0.937**
(-0.25, -0.25)	R1	0.000	0.058	0.144	0.039	0.222	0.777	0.854	0.911
	R2	**0.013**	**0.133**	0.117	**0.054**	**0.242**	**0.798**	**0.902**	**0.939**
$T = 100$		$H_a : d > d_0$				$H_a : d < d_0$			
ϕ	St./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
(0.25, 0.25)	R1	0.017	0.124	0.103	0.056	0.094	0.126	0.285	0.537
	R2	**0.182**	**0.148**	**0.150**	**0.092**	**0.110**	**0.131**	**0.297**	**0.596**
(0.25, -0.25)	R1	0.107	0.150	0.100	0.076	0.065	0.264	0.853	0.860
	R2	**0.583**	**0.192**	**0.109**	**0.088**	**0.098**	**0.273**	0.689	**0.906**
(-0.25, 0.25)	R1	0.006	0.105	0.120	0.052	0.208	0.718	0.943	0.986
	R2	**0.142**	**0.212**	**0.157**	**0.077**	**0.209**	**0.719**	**0.960**	**0.994**
(-0.25, -0.25)	R1	0.002	0.105	0.164	0.078	0.304	0.830	0.966	0.986
	R2	**0.064**	**0.165**	**0.191**	0.072	**0.307**	**0.835**	**0.984**	**0.994**
$T = 300$		$H_a : d > d_0$				$H_a : d < d_0$			
(ϕ_1, ϕ_2)	St./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
(0.25, 0.25)	R1	0.115	0.193	0.032	0.102	0.167	0.630	0.972	1.000
	R2	**0.361**	**0.476**	**0.091**	**0.103**	**0.254**	**0.633**	**0.976**	1.000
(0.25, -0.25)	R1	0.100	0.119	0.412	0.514	0.849	1.000	1.000	1.000
	R2	**0.325**	**0.295**	**0.591**	**0.531**	0.847	1.000	1.000	1.000
(-0.25, 0.25)	R1	0.966	0.231	0.004	0.112	0.463	0.975	1.000	1.000
	R2	**0.981**	**0.400**	**0.059**	**0.121**	0.453	**0.977**	1.000	1.000
(-0.25, -0.25)	R1	0.081	0.116	0.740	0.715	0.921	1.000	1.000	1.000
	R2	**0.355**	**0.305**	**0.804**	0.713	0.918	1.000	1.000	1.000
$T = 1000$		$H_a : d > d_0$				$H_a : d < d_0$			
(ϕ_1, ϕ_2)	St./d_0	0.00	0.25	0.50	0.75	1.25	1.50	1.75	2.00
(0.25, 0.25)	R1	0.554	0.609	0.711	0.402	0.467	0.903	1.000	1.000
	R2	**0.708**	**0.777**	**0.734**	**0.408**	**0.499**	**0.907**	1.000	1.000
(0.25, -0.25)	R1	0.500	0.554	0.709	0.663	0.998	1.000	1.000	1.000
	R2	**0.598**	**0.695**	0.709	**0.678**	0.998	1.000	1.000	1.000
(-0.25, 0.25)	R1	0.998	0.733	0.807	0.611	0.963	1.000	1.000	1.000
	R2	0.998	**0.688**	**0.909**	**0.708**	**0.977**	1.000	1.000	1.000
(-0.25, -0.25)	R1	0.766	0.723	0.831	0.787	0.950	1.000	1.000	1.000
	R2	0.721	**0.911**	**0.879**	0.787	0.950	1.000	1.000	1.000

Those values where the rejection frequencies corresponding to R2 are superior to those of R1 appear in bold type.

will be $H_a : d > d_0$), and $d = 1.25$, 1.50, 1.75 and 2 (with $H_a : d < d_0$). Starting with $T = 70$, we see that R2 outperforms R1 in all cases. If $T = 100$, there are only two cases where R1 produces higher rejection frequencies than R2. If $T = 300$, the values are very similar in both cases when directed against $H_a : d < d_0$, while R2 outperforms R1 if the alternatives are of form: $d > d_0$. Finally, if $T = 1000$, the results are very similar, and R2 is superior to R1 in only 8 out of the 32 cases presented. The local efficiency property of the tests seems to assert itself in view of the fact that the rejection frequencies are relatively high even for small departures from the null. Thus, for example, if $T \geq 300$, the rejection probabilities corresponding to $d = 0.75$ and 1.25 are higher than 0.9 in practically all cases. The only striking feature of these results occurs at $d = 0$, where the rejection frequencies are low and smaller than at $d = 0.25$. In other words, we do not observe a monotonic increase in the rejection frequencies as we move from $d_0 = 0.75$ through 0. This may be related to the fact that the AR coefficients are Yule-Walker estimates and thus, though they are automatically smaller than 1 in absolute value, they can be arbitrarily close to 1 and then, they might be competing with d in describing the nonstationarity of the series. Note, for example, that if the true process contains a unit root and we test for $I(0)$ with AR(1) disturbances, the procedure estimates the AR coefficient with a value extremely close to 1 and thus, it might not reject the null hypothesis of $d = 0$. Table 5 assumes that the true model contains AR(2) disturbances with $(\phi_1, \phi_2) = (0.25, 0.25); (0.25, -0.25); (-0.25, 0.25)$ and $(-0.25, -0.25)$. We again observe here a gain in efficiency when R2 is employed, especially if T is small. Similarly to the previous table, the rejection probabilities of R1 are in some cases small if $d_0 = 0$, and they improve with the new version of the test.

4 An empirical application

The time series data analyzed in this section correspond to the monthly observations of the Spanish Consumer Price Index (CPI), seasonally adjusted, for the time period 1970m1-1997m12. Denoting the time series by y_t, we employ the model given by (2.1) and (1.3) throughout with $z_t = (1, t)'$, $t \geq 1$, $z_t = (0, 0)'$ otherwise, i.e.,

$$y_t = \alpha + \beta t + x_t \ , \quad t = 1, 2, \ldots, \tag{4.1}$$

$$(1 - L)^d x_t = u_t \ , \quad t = 1, 2, \ldots, \tag{4.2}$$

modelling the $I(0)$ disturbances u_t as white noise and AR(1) and AR(2) process. Higher order autoregressions were also employed and the results were very similar to those reported here for the AR(1) and AR(2) cases.

We start with the assumption that u_t in (4.2) is white noise. Clearly in this case, R1 and R2 coincide, and we report in Table 6 the values of the one-sided statistic $\hat{r}$ in (2.8), testing H_0 (2.2) for values of $d_0 = 0.50$ (0.10), 1.50. Thus, we test for stationarity ($d = 0.50$), for a unit root ($d = 1$) as well as other fractionally integrated possibilities. A notable feature observed in this table is that the value of the test statistic monotonically decreases as d_0 increases. This monotonic decrease

is to be expected given that it is a one-sided statistic. Thus, for example, if H_0 (2.2) is rejected with $d = 0.70$ against $d > 0.70$, an even more significant result in this direction should be expected when $d = 0.60$ or $d = 0.50$ are tested. We also see that the only non-rejection value in this context of white noise disturbances corresponds to $d = 1.1$, and the unit root null is rejected against higher orders of integration. Allowing for weakly autocorrelated disturbances, we report the results based on the two statistics, i.e., R1 and R2. Starting with R1, there is a lack of monotonic decrease in the value of the test statistic for small values of d_0. We can argue (as in Gil-Alana and Robinson, 1997) that this is due to a problem of misspecification. In general, misspecification inflates both numerator and denominator of $\hat{r}$ to varying degrees, and thus affects $\hat{r}$ in a complicated way. However, as mentioned in Section 3, it may also be due to the fact that the AR coefficients are Yule- Walker estimates, entailing roots that are automatically less than one in absolute value but that can be arbitrarily close to one and thus, they might be competing with the fractional differencing parameter in describing the nonstationary component of the series. Computing R2 for the same range of d_0-values, we see that this problem is eliminated. Thus, the reason for the lack of monotonicity in R1 might be explained by the fact that the regression coefficients are not precisely estimated by R1, affecting the estimated AR coefficients through the residuals $\tilde{u}_t$ in (2.12). In fact, we see that performing R2 for both types of AR disturbances, the values of the test statistic monotonically decrease with d_0, and the non-rejection values occur now when d = 1.20 with AR(1) disturbances and when $d = 1.20$ and 1.30 with AR(2)u_t.

Table 6 *Testing H_0 (2.2) in (4.1) and (4.2) with R1 and R2 in the Spanish C.P.I.*

		Values of d_0										
u_t	Stat.	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50
White noise	R1	45.06	43.15	37.48	27.79	16.75	7.61	**1.55**	-2.00	-4.02	-5.19	-5.89
AR(1)	R1	-0.88	1.00	7.51	14.79	13.15	8.40	3.46	-0.30	-2.73	-4.17	-4.99
	R2	21.59	18.38	17.13	16.93	13.56	8.44	3.46	**-0.30**	-2.73	-4.17	-4.98
AR (2)	R1	-3.48	-2.30	-0.59	2.93	6.35	6.65	4.39	1.46	-0.98	-2.62	-3.60
	R2	7.49	7.41	7.38	6.73	6.56	6.05	4.39	**1.48**	**-0.93**	-2.56	-3.55

The non-rejection values at the 5% significance level appear in bold type

We can conclude the analysis in this section by saying that the series corresponding to the monthly observations of the Spanish CPI is clearly nonstationary, with an order of integration higher than 1, implying that its first differences still possesses a component of long memory behaviour. This result, however, should be taken with caution. Spanish data from 1979 to 1997 may have suffered different regime shifts, and this has been found in recent years to be crucial in the determination of the long memory parameter. (See, e.g., Diebold and Inoue, 2001, Gourieroux and Jasiak, 2001; etc.).

5 Conclusions

In this article we have shown that when testing fractionally integrated hypotheses in the presence of deterministic regressors and weakly autocorrelated disturbances, Robinson's test (1994) has better statistical properties if the regression coefficients and the residuals are estimated by GLS rather than OLS. Several Monte Carlo experiments conducted throughout the paper show that the test improves when GLS is employed. Furthermore, in an empirical application (using monthly data of the Spanish CPI), we show that the lack of monotonicity observed in the value of the one-sided statistic with respect to the order of integration (in case of AR disturbances) is eliminated when using the new version of the test.

It might also be worthwhile to study the possibility of misspecification with respect to the short-run components of the series. However, given that Robinson's (1994) procedure is purely parametric, the best results would be clearly obtained when we correctly specify u_t. In Robinson (1994) several Monte Carlo experiments were conducted comparing his test with other more classical unit-root tests (Dickey and Fuller, 1979, Phillips and Perron, 1988; etc.). He showed that his test outperforms the others in practically all cases if the alternatives are of fractional form. In that respect the new GLS version improves the finite-sample behaviour of Robinson (1994) and thus, it should be preferred to the other methods. On the other hand, there exist several semiparametric methods of estimating and testing the fractional differencing parameter (e.g., Geweke and Porter-Hudak, 1983; Robinson, 1995a, b; etc.). These methods, however, will be clearly less efficient, under correct specification, than Robinson's (1994) parametric one, and they might be too sensitive to the choice of the bandwidth parameter numbers.

This article can be extended in several other directions. For example, the optimality of the LM tests crucially depends on the normality assumption and therefore it would be interesting to see what happens when the disturbances are asymmetric and/or fat tailed. Also, we could have employed other more intuitive representations of the simulation results (e.g., Davidson and MacKinnon, 1998), extending the analysis to cover the case, for example, of ARMA processes, with the AR and MA roots being fairly close to the nonstationary and non- invertibility regions respectively. However, the main goal in this paper was to show the gain in efficiency when using GLS rather than OLS throughout a simple fractional autoregressive model. Work on the above approaches is now under progress.

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