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A Φ_1–beta distribution

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Abstract: A new distribution which contains several of the known generalizations of the beta distribution as particular cases is introduced. Various structural properties of this distribution are derived, including its cdf, moment generating function, characteristic function, moments, mean deviation about the mean, mean deviation about the median, entropy, asymptotic distribution of extreme order statistics, maximum likelihood estimates and the Fisher information matrix. An application to consumer price indices is illustrated to show that the proposed distribution is a better model to economic data than one based on the standard beta distribution.

Key words: Beta distribution, economic data.

1 Introduction

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see for example Barlow and Proschan (1975).

A random variable X is said to have the standard beta distribution with parameters α and β if its probability density function (pdf) is:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \tag{1.1}$$

for $0 < x < 1$, $\alpha > 0$ and $\beta > 0$, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

denotes the beta function. Many generalizations of (1.1) involving algebraic, exponential and hypergeometric functions have been proposed in the literature; see Chapter 25 in Johnson *et al.* (1995) and Gupta and Nadarajah (2004) for detailed accounts. In this note, we introduce a new distribution that contains several of the known generalizations as particular cases. We derive various structural properties of this new distribution, including its cdf, moment generating function, characteristic function, moments, mean deviation about the mean, mean deviation about the median, entropy, asymptotic distribution of extreme order statistics, maximum likelihood estimates and the Fisher information matrix. We also present an application of the proposed model to consumer price indices.

The calculations of this paper involve several special functions, including the degenerate hypergeometric series of two variables defined by

$$\Phi_1(a, b, c, x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_n x^m y^n}{(c)_{m+n} m! n!},$$

the degenerate hypergeometric function of two variables defined by

$$F_1(a, b, c; d; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m (c)_n x^m y^n}{(d)_{m+n} m! n!},$$

the confluent hypergeometric function of one variable defined by

$${}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

the Gauss hypergeometric function of one variable defined by

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}$$

and, the Kummer function defined by

$$\Psi(a, b; x) = \frac{\Gamma(1-b)\ {}_1F_1(a; b; x)}{\Gamma(1+a-b)} + \frac{\Gamma(b-1) x^{1-b}\ {}_1F_1(1+a-b; 2-b; x)}{\Gamma(a)},$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. We also need the following important lemmas.

Lemma 1. *(Equation (2.2.6.1), Prudnikov et al., 1986, Volume 1) For* $\alpha > 0$ *and* $\beta > 0$*,*

$$\begin{aligned} & \int_a^b (x-a)^{\alpha-1}(b-x)^{\beta-1}(cx+d)^{\gamma} dx \\ = \ & (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma} B(\alpha, \beta)\ {}_2F_1\left(\alpha, -\gamma; \alpha+\beta; \frac{c(a-b)}{ac+d}\right). \end{aligned}$$

Lemma 2. *(Equation (2.2.8.5), Prudnikov et al., 1986, Volume 1) For* $a > 0$*,* $\alpha > 0$ *and* $\beta > 0$*,*

$$\begin{aligned} \int_0^a x^{\alpha-1}(a-x)^{\beta-1}(1-ux)^{-\rho}(1-vx)^{-\lambda} dx = \ & a^{\alpha+\beta-1} B(\alpha, \beta) \\ & F_1(\alpha, \rho, \lambda, \alpha+\beta; ua, va). \end{aligned}$$

Lemma 3. *(Equation (2.3.6.9), Prudnikov et al., 1986, volume 1) For* $\alpha > 0$ *and* $p > 0$,

$$\int_0^a x^{\alpha-1}(x+z)^{-\rho}\exp(-px)dx = \Gamma(\alpha)z^{\alpha-\rho}\Psi\left(\alpha,\alpha+1-\rho;pz\right).$$

Lemma 4. *(Equation (2.3.8.1), Prudnikov et al., 1986, Volume 1) For* $a > 0$, $\alpha > 0$ *and* $\beta > 0$,

$$\begin{aligned}\int_0^a x^{\alpha-1}(a-x)^{\beta-1}(x+z)^{-\rho}\exp(-px)dx &= B(\alpha,\beta)z^{-\rho}a^{\alpha+\beta-1}\\ &\quad \Phi_1\left(\alpha,\rho,\alpha+\beta;-a/z,ap\right).\end{aligned}$$

Further properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Probability density function

We define the new distribution by the pdf

$$f(x) = \frac{Cx^{\alpha-1}(1-x)^{\beta-1}\exp(-px)}{(x+z)^{\rho}} \tag{2.1}$$

for $0 < x < 1$, $\alpha > 0$, $\beta > 0$, $\rho \geq 0$, $p \geq 0$ and $-\infty < z < \infty$, where C denotes the normalizing constant. Note that if $z = 0$ then one must have $\alpha > \rho$. Application of Lemma 4 shows that the normalizing constant is given by

$$\frac{1}{C} = B(\alpha,\beta)z^{-\rho}\Phi_1\left(\alpha,\rho,\alpha+\beta;-\frac{1}{z},p\right)$$

and thus we refer to (2.1) as the Φ_1 *beta distribution.* This new distribution is very flexible and it contains several of the known generalizations of (1.1) as particular cases. The standard beta distribution is the particular case for $p = 0$ and $\rho = 0$; Libby and Novick (1982)'s generalized beta distribution is the particular case for $p = 0$ and $\rho = \alpha + \beta$; the Gauss hypergeometric distribution due to Armero and Bayarri (1994) is the particular case for $p = 0$; and, the confluent hypergeometric distribution due to Gordy (1998a) is the particular case for $\rho = 0$.

A location-scale transformation of (2.1) has been proposed earlier in an unpublished paper by Gordy (1998b). In terms of mathematical properties, the paper by Gordy (1998b) only provides the mgf and the moments. The mathematical treatment of (2.1) given in our paper is much more comprehensive.

Let us now consider the shape of (2.1). The first derivative of $\log f$ is

$$\frac{d\log f}{dx} = \frac{\alpha-1}{x} - \frac{\beta-1}{1-x} - \frac{\rho}{x+z} - p.$$

Setting this to zero, one obtains the cubic equation

$$\begin{aligned} & px^3 - \{\alpha + \beta + (1-z)p - \rho\} x^2 - \{(1-\alpha)(1-z) + (\beta - 1)z + pz + \rho\} x \\ + & (\alpha - 1)z = 0. \end{aligned}$$

Thus, in principle, the pdf can contain up to three turning points. Some possible shapes are illustrated in Figure 1 for selected values of α, β, z and ρ. Note that several of the curves contain more than one turning point.

3 Cumulative distribution function

Direct expressions for the cdf of (2.1) are not possible. Here, we provide two series representations for the cdf. Firstly, using the series expansion

$$(1-x)^{\beta-1} = \sum_{j=0}^{\infty} \binom{\beta-1}{j} (-x)^j, \tag{3.1}$$

one can write

$$\begin{aligned} F(x) &= C \int_0^x \frac{y^{\alpha-1}(1-y)^{\beta-1} \exp(-py)}{(y+z)^\rho} dy \\ &= C \int_0^x \frac{y^{\alpha-1} \exp(-py)}{(y+z)^\rho} \left\{ \sum_{j=0}^{\infty} \binom{\beta-1}{j} (-y)^j \right\} dy \\ &= C \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} \int_0^x \frac{y^{j+\alpha-1} \exp(-py)}{(y+z)^\rho} dy \\ &= C z^{-\rho} \sum_{j=0}^{\infty} \frac{(-1)^j x^{\alpha+j}}{\alpha+j} \binom{\beta-1}{j} \Phi_1 \left(\alpha+j, \rho, \alpha+j+1; -\frac{x}{z}, px \right), \end{aligned} \tag{3.2}$$

where the last step follows by application of Lemma 4. Note that the infinite sum in (3.2) will reduce to a finite sum if β is an integer. Secondly, using the series expansion

$$\exp(-px) = \sum_{j=0}^{\infty} \frac{(-px)^j}{j!}, \tag{3.3}$$

one can write

$$F(x) = C \int_0^x \frac{y^{\alpha-1}(1-y)^{\beta-1}}{(y+z)^\rho} \left\{ \sum_{j=0}^{\infty} \frac{(-py)^j}{j!} \right\} dy$$

$$
\begin{aligned}
&= C\sum_{j=0}^{\infty}\frac{(-p)^j}{j!}\int_0^x \frac{y^{\alpha+j-1}(1-y)^{\beta-1}}{(y+z)^{\rho}}dy \\
&= Cz^{-\rho}x^{\alpha}\sum_{j=0}^{\infty}\frac{(-p)^j}{j!(\alpha+j)}F_1\left(\alpha+j,1-\beta,\rho,\alpha+j+1;x,-\frac{x}{z}\right), \quad (3.4)
\end{aligned}
$$

where the last step follows by application of Lemma 2.

4 Characteristic function

Here, we derive the moment generating and the characteristic functions of a random variable X with pdf (2.1). The moment generating function (mgf) is defined by $M(t) = \mathrm{E}(\exp(tX))$. It can be calculated easily as

$$
\begin{aligned}
M(t) &= C\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta-1}\exp\{-(p-t)x\}}{(x+z)^{\rho}}dx \\
&= CB(\alpha,\beta)z^{-\rho}\Phi_1\left(\alpha,\rho,\alpha+\beta;-\frac{1}{z},p-t\right),
\end{aligned}
$$

where we have applied Lemma 4. Thus, the characteristic function of X defined by $\phi(t) = \mathrm{E}(\exp(itX))$ takes the form

$$
\phi(t) = CB(\alpha,\beta)z^{-\rho}\Phi_1\left(\alpha,\rho,\alpha+\beta;-\frac{1}{z},p-it\right),
$$

where $i = \sqrt{-1}$ is the complex number.

5 Moments

The nth moment of a random variable X with pdf (2.1) can be calculated easily as

$$
\begin{aligned}
E\left(X^n\right) &= C\int_0^1 \frac{x^{n+\alpha-1}(1-x)^{\beta-1}\exp\left(-px\right)}{(x+z)^{\rho}}dx \\
&= CB(\alpha,\beta)z^{-\rho}\Phi_1\left(\alpha+n,\rho,\alpha+\beta+n;-\frac{1}{z},p\right),
\end{aligned}
$$

where we have applied Lemma 4. One can also derive two series representations for the nth moment similar to those in Section 3. Firstly, using the expansion (3.1), one can write

$$
E\left(X^n\right) = C\int_0^1 \frac{x^{n+\alpha-1}(1-x)^{\beta-1}\exp(-px)}{(x+z)^{\rho}}dx
$$

$$
\begin{aligned}
&= C\int_0^1 \frac{x^{n+\alpha-1}\exp(-px)}{(x+z)^\rho}\left\{\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-x)^j\right\}dx \\
&= C\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}\int_0^1 \frac{x^{j+n+\alpha-1}\exp(-px)}{(x+z)^\rho}dx \\
&= C\sum_{j=0}^{\infty}\binom{\beta-1}{j}(-1)^j z^{\alpha+j+n-\rho}\Gamma(\alpha+j+n) \\
&\quad \Psi\left(\alpha+j+n,\alpha+j+n+1-\rho;pz\right), \qquad (5.1)
\end{aligned}
$$

where the last step follows by application of Lemma 3.

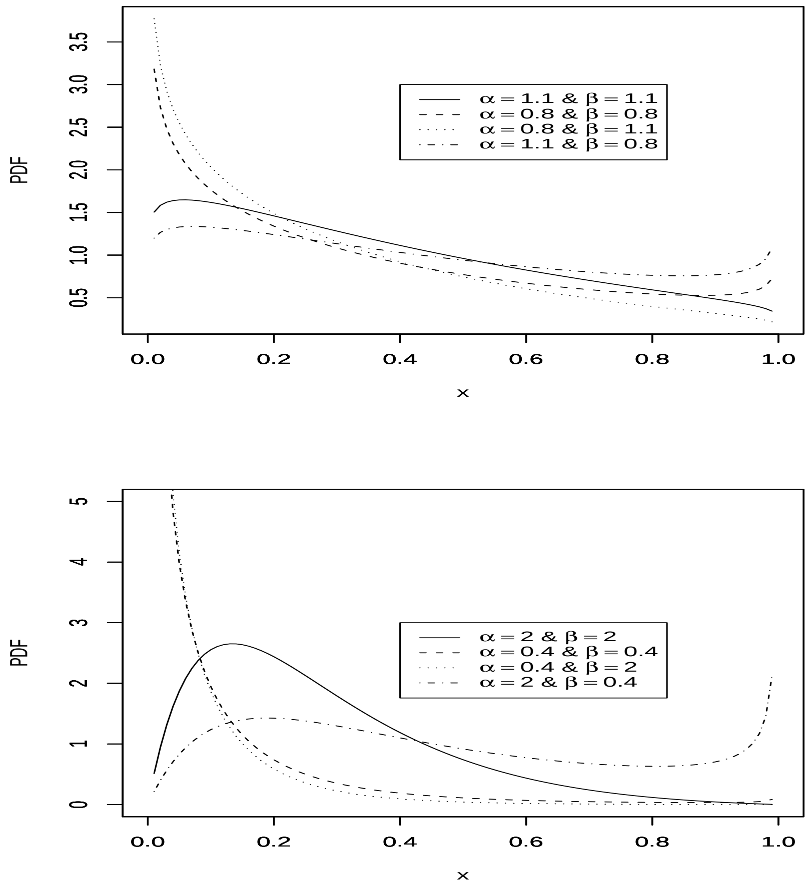

Figure 1 *Plots of the pdf (2.1) for $p = 1$, $z = 1$, $\rho = 1$ (top) and $p = 1$, $z = 1$, $\rho = 6$ (bottom).*

Note that the infinite sum in (5.1) will reduce to a finite sum if β is an integer. Secondly, using the expansion (3.3), one can write

$$\begin{aligned}
E\left(X^n\right) &= C \int_0^1 \frac{x^{n+\alpha-1}(1-x)^{\beta-1}}{(x+z)^{\rho}} \left\{ \sum_{j=0}^{\infty} \frac{(-px)^j}{j!} \right\} dx \\
&= C \sum_{j=0}^{\infty} \frac{(-p)^j}{j!} \int_0^1 \frac{x^{n+\alpha+j-1}(1-x)^{\beta-1}}{(x+z)^{\rho}} dx \\
&= C z^{-\rho} \sum_{j=0}^{\infty} \frac{(-p)^j B(n+\alpha+j,\beta)}{j!} \\
&\quad {}_2F_1\left(n+\alpha+j, \rho; n+\alpha+\beta+j; -\frac{1}{z}\right), \qquad (5.2)
\end{aligned}$$

where the last step follows by application of Lemma 1.

6 Mean deviations

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and the median. These are known as the mean deviation about the mean and the mean deviation about the median – defined by

$$\delta_1(X) = \int_0^1 |x-\mu| \, f(x) dx$$

and

$$\delta_2(X) = \int_0^1 |x-M| \, f(x) dx,$$

respectively, where $\mu = \mathrm{E}(X)$ and $M = \mathrm{Median}(X)$. These measures can be calculated using the relationships

$$\begin{aligned}
\delta_1(X) &= \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^1 (x-\mu) f(x) dx \\
&= 2 \int_{\mu}^1 (x-\mu) f(x) dx \\
&= 2 \int_{\mu}^1 x f(x) dx - 2\mu \left\{1 - F(\mu)\right\} \\
&= 2\mathrm{E}(X) - 2 \int_0^{\mu} x f(x) dx - 2\mu \left\{1 - F(\mu)\right\} \qquad (6.1)
\end{aligned}$$

and

$$\begin{aligned}\delta_2(X) &= \int_0^M (M-x) f(x) dx + \int_M^1 (x-M) f(x) dx \\ &= MF(M) - M\left\{1-F(M)\right\} - \int_0^M xf(x)dx + \int_M^1 xf(x)dx \\ &= 2\int_0^M xf(x)dx - \mathrm{E}(X).\end{aligned} \tag{6.2}$$

Thus, calculating $\delta_1(X)$ and $\delta_2(X)$ amounts to calculating $\int_0^a xf(x)dx$. Applying the series expansions (3.1) and (3.3) in the same way as in Sections 3 and 5, one can obtain the two representations:

$$\begin{aligned}\int_0^a xf(x)dx &= Cz^{-\rho} \sum_{j=0}^{\infty} \frac{(-1)^j a^{\alpha+j+1}}{\alpha+j+1} \binom{\beta-1}{j} \\ & \quad \Phi_1\left(\alpha+j+1, \rho, \alpha+j+2; -\frac{a}{z}, pa\right)\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}\int_0^a xf(x)dx &= Cz^{-\rho} \sum_{j=0}^{\infty} \frac{(-p)^j a^{\alpha+j+1}}{j!(\alpha+j+1)} \\ & \quad F_1\left(\alpha+j+1, 1-\beta, \rho, \alpha+j+2; a, -\frac{a}{z}\right).\end{aligned} \tag{6.4}$$

Expressions for the mean deviations follow by substituting (6.3)–(6.4) into (6.1) and (6.2).

7 Rényi entropy

An entropy of a random variable X is a measure of variation of the uncertainty. Rényi entropy is defined by

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log\left\{\int f^\gamma(x)dx\right\},$$

where $\gamma > 0$ and $\gamma \neq 1$ (Rényi, 1961). It follows easily by application of Lemma 4 that

$$\begin{aligned}& \int_0^1 f^\gamma(x)dx \\ &= C^\gamma \int_0^1 \frac{x^{\gamma\alpha-\gamma}(1-x)^{\gamma\beta-\gamma}\exp(-p\gamma x)}{(x+z)^{\gamma\rho}} dx \\ &= C^\gamma B\left(\alpha\gamma-\gamma+1, \beta\gamma-\gamma+1\right) z^{-\rho\gamma} \\ & \quad \Phi_1\left(\alpha\gamma-\gamma+1, \rho\gamma, \alpha\gamma+\beta\gamma-2\gamma+2; -\frac{1}{z}, p\gamma\right).\end{aligned}$$

Thus, Rényi entropy for (2.1) is given by

$$\begin{aligned}\mathcal{J}_R(\gamma) &= \frac{1}{1-\gamma}\Bigg\{\gamma\log C + \log B\left(\alpha\gamma-\gamma+1,\beta\gamma-\gamma+1\right) - \rho\gamma\log z \\ &\quad + \log\Phi_1\left(\alpha\gamma-\gamma+1,\rho\gamma,\alpha\gamma+\beta\gamma-2\gamma+2;-\frac{1}{z},p\gamma\right)\Bigg\}.\end{aligned}$$

8 Asymptotics

If $X_1,\ldots,X_n$ is a random sample from (2.1) and if $\bar{X}=(X_1+\cdots+X_n)/n$ denotes the sample mean then by the usual central limit theorem $\sqrt{n}(\bar{X}-\mathrm{E}(X))/\sqrt{\mathrm{Var}(X)}$ approaches the standard normal distribution as $n\to\infty$. Sometimes one would be interested in the asymptotics of the extreme values $M_n=\max(X_1,\ldots,X_n)$ and $m_n=\min(X_1,\ldots,X_n)$. Note from (2.1) that $f(t)\sim Cz^{-\rho}t^{\alpha-1}$ as $t\to 0$ and $f(t)\sim C\exp(-p)(1+z)^{-\rho}(1-t)^{\beta-1}$ as $t\to 1$. Thus, it follows by using L'Hospital's rule that

$$\frac{1-F(1-xh)}{1-F(1-h)} \quad\to\quad x^{\beta}$$

and

$$\frac{F(xh)}{F(h)} \quad\to\quad x^{\alpha}$$

as $h\to 0$. Hence, it follows from Theorem 1.6.2 in Leadbetter *et al.* (1987) that there must be norming constants $a_n>0$, b_n, $c_n>0$ and d_n such that

$$\Pr\left\{a_n\left(M_n-b_n\right)\le x\right\} \quad\to\quad \exp\left\{-(-x)^{\beta}\right\}$$

and

$$\Pr\left\{c_n\left(m_n-d_n\right)\le x\right\} \quad\to\quad 1-\exp\left(-x^{\alpha}\right)$$

as $n\to\infty$.

9 Estimation

Here, we consider maximum likelihood estimation of the parameters when $X_1,\ldots,$ X_n is a random sample from (2.1) and also provide expressions for the associated Fisher information matrix. The log-likelihood is

$$\begin{aligned}\log L\left(\alpha,\beta,\rho,z,p\right) &= n\log C+(\alpha-1)\sum_{j=1}^{n}\log X_j+(\beta-1)\sum_{j=1}^{n}\log\left(1-X_j\right) \\ &\quad -\rho\sum_{j=1}^{n}\log\left(X_j+z\right)-p\sum_{j=1}^{n}X_j.\end{aligned}$$

The first derivatives with respect to the five parameters are:

$$\frac{\partial \log L}{\partial \alpha} = \sum_{j=1}^{n} \log X_j + \frac{n}{C} \frac{\partial C}{\partial \alpha},$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{j=1}^{n} \log \left(1 - X_j\right) + \frac{n}{C} \frac{\partial C}{\partial \beta},$$

$$\frac{\partial \log L}{\partial \rho} = -\sum_{j=1}^{n} \log \left(X_j + z\right) + \frac{n}{C} \frac{\partial C}{\partial \rho},$$

$$\frac{\partial \log L}{\partial z} = -\rho \sum_{j=1}^{n} \frac{1}{X_j + z} + \frac{n}{C} \frac{\partial C}{\partial z}$$

and

$$\frac{\partial \log L}{\partial p} = -\sum_{j=1}^{n} X_j + \frac{n}{C} \frac{\partial C}{\partial p}.$$

Thus, the maximum likelihood estimates of the five parameters are the solutions of the equations:

$$\frac{n}{C} \frac{\partial C}{\partial \alpha} = -\sum_{j=1}^{n} \log X_j,$$

$$\frac{n}{C} \frac{\partial C}{\partial \beta} = \sum_{j=1}^{n} \log \left(1 - X_j\right),$$

$$\frac{n}{C} \frac{\partial C}{\partial \rho} = \sum_{j=1}^{n} \log \left(X_j + z\right),$$

$$\frac{n}{C} \frac{\partial C}{\partial z} = \rho \sum_{j=1}^{n} \frac{1}{X_j + z}$$

and

$$\frac{n}{C} \frac{\partial C}{\partial p} = \sum_{j=1}^{n} X_j.$$

Calculation of the associated Fisher information matrix requires second-order derivatives of $\log L$. All of the second-order derivatives take the form

$$\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j} = -\frac{n}{C^2}\frac{\partial C}{\partial\theta_i}\frac{\partial C}{\partial\theta_j} + \frac{n}{C}\frac{\partial^2 C}{\partial\theta_i \partial\theta_j}$$

except for

$$\frac{\partial^2 \log L}{\partial\rho \partial z} = -\sum_{j=1}^{n} \frac{1}{X_j + z} - \frac{n}{C^2}\frac{\partial C}{\partial\rho}\frac{\partial C}{\partial z} + \frac{n}{C}\frac{\partial^2 C}{\partial\rho \partial z}$$

and

$$\frac{\partial^2 \log L}{\partial^2 z} = \rho\sum_{j=1}^{n} \frac{1}{(X_j + z)^2} - \frac{n}{C^2}\left(\frac{\partial C}{\partial z}\right)^2 + \frac{n}{C}\frac{\partial^2 C}{\partial z^2}.$$

Thus, the elements of the Fisher information matrix are straight-forward upon noting that

$$E\left[(X+z)^{-m}\right] = CB(\alpha,\beta) z^{-(\rho+m)} \Phi_1\left(\alpha, \rho+m, \alpha+\beta; -\frac{1}{z}, p\right),$$

which follows by the use of Lemma 4.

10 Application

We now illustrate an application of the proposed generalized beta distribution to consumer price index data. We collected the data on this index for the six countries: United States, United Kingdom, Japan, Canada, Germany and Australia. The data were extracted from the web-site http://www.globalfindata.com/ and the range of data for each country is shown in the table below.

Table 1 *Datasets.*

Country	Range of Data
Australia	1901 to 2003
Canada	1910 to 2003
Germany	1923 to 2003
Japan	1868 to 2003
United Kingdom	1800 to 2003
United States	1820 to 2003

Taking the ratio $W = X/(X+Y)$, we attempted to model the relative economic performance of each country against another over the range of overlapping years. This yields 15 data sets for the variable W. As expected, some of the data for

W appeared to concentrate to a subinterval of $[0, 1]$ and so suitable location–scale transformations were applied to make the data span from 0 to 1. For each data set, we fitted the standard beta distribution and the Φ_1 beta distribution (given by (1.1) and (2.1), respectively) by the method of maximum likelihood. The quasi-Newton algorithm `nlm` in the `R` software package (Dennis and Schnabel, 1983; Schnabel *et al*, 1985; Ihaka and Gentleman, 1996) was used to maximize the likelihood.

The results of the fits were remarkable. In each fit, the maximized log-likelihood for the Φ_1 beta model turned up significantly higher than that for the standard beta model. Here, we give details for just two of the 15 data sets:

- for the (United States, United Kingdom) data set $\log L = 5.145$ for the standard beta model and $\log L = 13.150$ for the Φ_1 beta model. The corresponding fitted densities superimposed with the empirical density are shown in Figure 2.

- for the (United States, Australia) data set the fitted estimates were $\log L = 1.372$ for the standard beta model and $\log L = 11.518$ for the Φ_1 beta model. The corresponding fitted densities superimposed with the empirical density are shown in Figure 3.

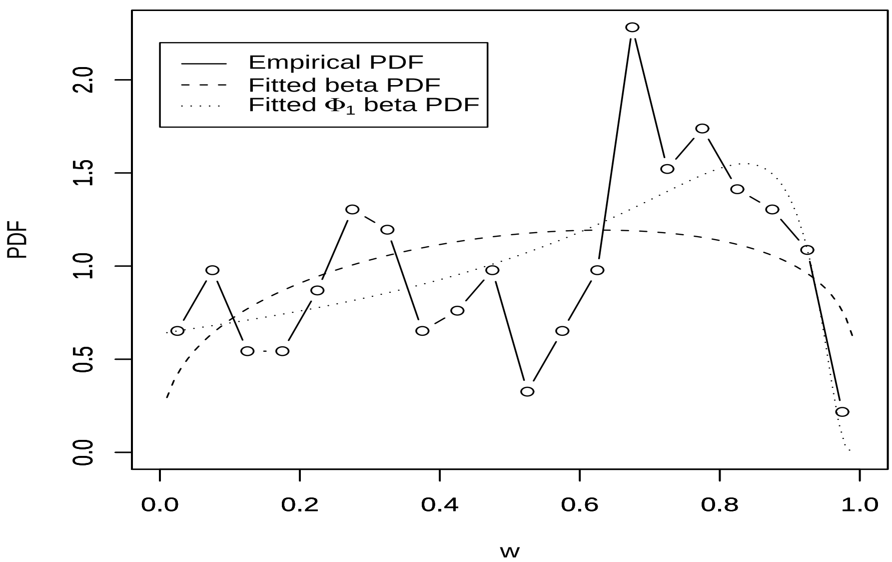

Figure 2 *The empirical and fitted densities for the consumer price indices of the United States and the United Kingdom.*

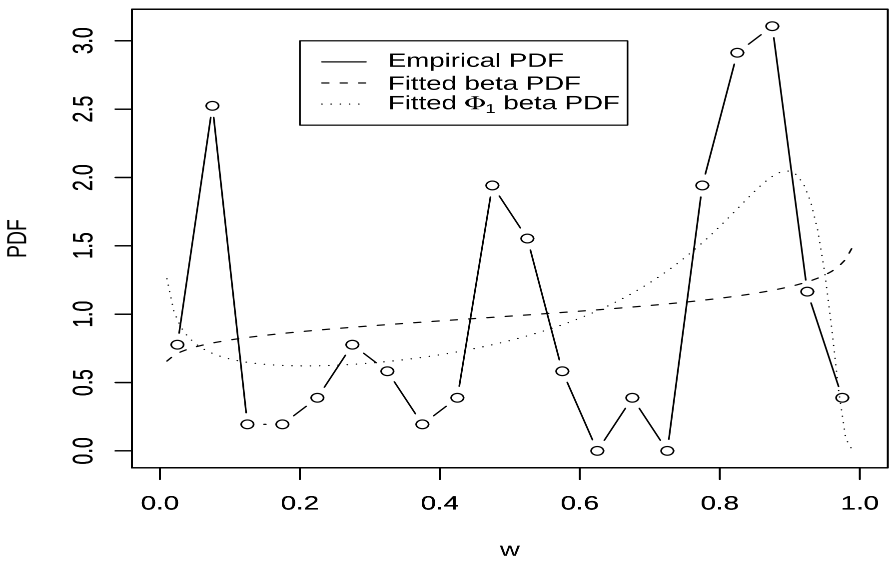

Figure 3 *The empirical and fitted densities for the consumer price indices of the United States and Australia.*

So, we can conclude at least in this situation that the Φ_1 beta model is better than one based on the standard beta distribution.

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