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Risk models with extremal subexponentiality

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Abstract: In this paper we consider risk models with a heavy-tailed parametric claim distribution from the subexponential class $\mathcal{S}$ with at least two parameters. We choose a proper convergence of a parameter, that makes the tail of the claims distribution heavier or lighter and then tend it to its limitation. Finally we proceed to an appropriate functional normalization in order to keep the distributional properties.

Key words: Cornish-Fisher expansion, normal approximation, Poisson distribution, u chart.

1 Introduction

In this paper the following problem is investigated: We consider a heavy-tailed parametric distribution from the subexponential class $\mathcal{S}$ with at least two parameters. We shall demand such a relation between the parameters, that the safety loading coefficient remains fixed. Choose a proper convergence of a parameter, that makes the tail of the claims distribution heavier and then tend it to its limitation. What happens then with the corresponding ruin probability under some special risk models?

We consider the classical risk model where the claims occur at random epochs which form a homogeneous Poisson process $\{N(t)\,,\ t \geq 0\}$ with intensity $\lambda > 0$, which is independent of the claims $Z_k\,,\ k = 1, 2, \ldots$. Using the notation of $\overline{B}(x) = 1 - B(x)$ for the tail of the claim distriburion $B(x)$, of $b(x)$ for the density and of $b = \mathbf{E}Z$ for the mean claim size, we denote the expected claim per time unit by $\rho = \lambda b$ and the

$$X(t) = \sum_{k=1}^{N(t)} Z_k$$

is the compound Poisson process representing the total claim amount accumulated until time t. Thus

$$F(x) = \frac{1}{b} \int_0^x \overline{B}(z)\, dz \tag{1.1}$$

is the integrated tail of the claim size distribution. In the classical risk model, the

Pollaczeck-Khinchine formula takes the form

$$\psi(u) = \Big(1 - \frac{\rho}{c}\Big) \sum_{n=0}^{\infty} \Big(\frac{\rho}{c}\Big)^n \overline{F}^{n*}(u)\,, \tag{1.2}$$

and gives the main tool for calculation of the ruin probability.

If $F \in \mathcal{S}$, the following asymptotic formula was found in Embrechts and Veraverbeke (1982)

$$\psi(u) \sim \frac{\rho}{c-\rho}\overline{F}(u) = \frac{\rho}{(c-\rho)\, b} \int_u^{\infty} \overline{B}(y)dy\,, \tag{1.3}$$

as $u \to \infty$.

The motivation of this problem comes from the following observation. In the vicinities of the critical values of the parameter, where the ergodicity condition does not hold any more, the ruin probability jumps to 1. The practical importance of this statement is shown through the unexpected result that the ruin probability on these neighborhoods does not depend any more on the initial capital. Obviously, this fact opens a new problem of approximation of the ruin probability in these areas.

Indeed, we often deal in insurance and finance with large claims that are described by heavy-tailed distributions (Pareto, Lognormal, Weibull, Loggamma, Burr). The known results reveal only asymptotic behavior of the ruin probabilities. Numerical calculations show that the accuracy of these asymptotic formulas can be quite low, especially in the range that is relevant for practical purposes (see for example Kalashnikov (1997) and Konstantinidis (1999)).

It is worthy of notice the special importance of heavy-tailed distributions, which is increasing the last years because of occasional appearance of huge claims. The problem consists in proposing other approximations that work in the area of practically significant values of the corresponding parameters and variables. To this end, the classification of the distributions describing large claims is promoted. This approach presents a new classification, since up to now all heavy-tailed distributions were considered as simple members of the subexponential class $\mathcal{S}$.

We concentrate our study on five concrete subexponential parametric families, widely used in insurance mathematics.

Example 1.1 *Pareto:*

$$\overline{B}(x) = \begin{cases} 1, & when \quad x \le k, \\ \left(\frac{k}{x}\right)^{\alpha}, & when \quad x > k, \end{cases}$$

with $\alpha > 1, \;\; k > 0$;

Example 1.2 *Lognormal:*

$$b(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\},$$

with μ *real number and* $\sigma > 0$*;*

Example 1.3 *Weibull:*

$$\overline{B}(x) = e^{-\nu^\tau x^\tau},$$

with $\nu > 0$, $0 < \tau < 1$;

Example 1.4 *Loggamma:*

$$b(x) = \frac{\alpha^p}{\Gamma(p)} \left[\ln(1+x)\right]^{p-1} (1+x)^{-\alpha-1},$$

with $\alpha > 0, \ \ p > 0$;

Example 1.5 *Burr:*

$$\overline{B}(x) = \left(\frac{\kappa}{\kappa + x^r}\right)^\alpha,$$

with $\kappa > 0, \ \ r > 0, \ \ \alpha > 1/r$;

2 The Heuristics

Firstly let us take the example of the Pareto distribution, in which the parameter α will be considered as its parameter of heavytailedness. If the other parameter k is fixed and α tends to its minimal value 1 and consequently the expectation of the claim sizes tends to infinity, and the corresponding ladder height process is not ergodic any more in the limit. In such a case the integrated tail claim distribution from (1.1) is meaningless and the Pollaczeck-Khinchine formula does not work. So in order to keep the balance within the chosen convergence of α, either the safety loading or in particular the mean claim

$$b = \frac{\alpha k}{\alpha - 1}$$

must be held fixed, which leads to the normalization. As a result of this, we put as our normalization condition

$$b = 1, \tag{2.1}$$

which is common for all the examples listed above. Thus we obtain $\rho = \lambda$.

In our example of the Pareto distribution the relation (2.1) implies that the second parameter k takes the value

$$k = \frac{\alpha - 1}{\alpha}.$$

It should be noted that this kind of ergodic control is not unique (see Kalashinikov (1997)).

Now the heavytailedness parameter converges to the limit which demonstrates its most heavily (superheavy) tailed distribution. Namely, in the example of the Pareto distribution, α tends to its least value as follows

$$\alpha \longrightarrow 1.$$

Secondly, repeating these steps for the Lognormal distribution, it follows that the normalization (2.1) implies

$$\mu = -\frac{\sigma^2}{2},$$

and that the value of the heavytailedness parameter σ tends as follows

$$\sigma \longrightarrow \infty,$$

caused by the need to identify the superheavy tailed distribution.

In the next example of the Weibull distribution, this pattern of the normalization (2.1) renders

$$\nu = \Gamma\left(1 + \frac{1}{\tau}\right),$$

where $\Gamma(.)$ denotes the Gamma function. The most heavy tailed distribution arises when the heavytailedness parameter τ tends to 0.

Further, when we consider the Loggamma distribution according to the normalization procedure (2.1), we take

$$\alpha = \frac{2^{1/p}}{2^{1/p} - 1}.$$

Now the heavytailedness parameter p tends to 0.

Finally, in the case of the Burr distribution the normalization (2.1) leads to

$$\kappa = \left(\frac{r\Gamma(\alpha)}{\Gamma(1/r)\Gamma(\alpha - 1/r)}\right)^r. \tag{2.2}$$

When r is fixed, the parameter of heavytailedness α tends to $1/r$.

Remark 2.1 If $B(x)$ belongs to the Pareto, Lognormal, Weibull, Loggamma or Burr distribution family, then for any $x > 0$ its tail tends to zero

$$\overline{B}(x) \to 0, \tag{2.3}$$

as the heavytailedness parameter tends to its limit ($\alpha \to 1$, $\sigma \to \infty$, $\tau \to 0$, $p \to 0$, $\alpha \to 1/r$ correspondingly).

Indeed, for the Pareto distribution family, we take:

$$\overline{B}(x) \sim \frac{1}{x}(\alpha - 1)^{\alpha} \to 0,$$

as $\alpha \to 1$ and for any $x > 0$.

For the Lognormal distribution family, it holds:

$$\overline{B}(x) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln x}{\sigma} + \frac{\sigma}{2}}^{\infty} \exp\left[-y^2/2\right] dy \to 0,$$

as $\sigma \to \infty$ and for any $x > 0$.

For the Weibull distribution family by using Stirling's formula it can be found that:

$$(\nu x)^{\tau} \to \left[x\Gamma\left(1+\frac{1}{\tau}\right)\right]^{\tau} \sim (\sqrt{2\pi})^{\tau}\left(\frac{1}{\tau}\right)^{\tau/2}\frac{x^{\tau}}{\tau e} \to \infty,$$

as $\tau \to 0$, from which the limit follows immediately.

For the Loggamma distribution family for any $\varepsilon \in (0, x)$, the following sequence of relations is true:

$$\overline{B}(x) = \frac{1}{\Gamma(p)}\int_{A_p(x)}^{\infty} w^{p-1}e^{-w}dw \leq \frac{1}{\Gamma(p)}\int_{\ln(1+x)}^{\infty} w^{p-1}e^{-w}dw \leq$$
$$\leq 1 - \frac{1}{\Gamma(p)(1+\varepsilon)}\int_0^{\ln(1+x)} w^{p-1}dw = 1 - \frac{\ln^p(1+x)}{p\Gamma(p)(1+\varepsilon)} \to \frac{\varepsilon}{1+\varepsilon},$$

as $p \to 0$, where

$$A_p(x) = \frac{2^{1/p}}{2^{1/p}-1}\ln(x+1)\,.$$

In the last step we considered the well-known asymptote

$$\Gamma\left(\alpha\right) \sim \frac{1}{\alpha}, \tag{2.4}$$

as $\alpha \to 0$.

For the Burr distribution family for fixed r the asymptote (2.4) gives:

$$\kappa \sim r^r\left(\alpha - \frac{1}{r}\right)^r \to 0,$$

as $\alpha \to 1/r$, from which the limit follows immediately.

3 Superheavy subexponential tails

Lemma 3.1 (Tsitsiashvili and Konstantinides (2001)) *In the classical risk model, if $B(x)$ belongs to the Pareto, Lognormal, Weibull, Loggamma or Burr distribution family, then for any $u > 0$, the ruin probability tends to a constant*

$$\psi(u) \to \frac{\rho}{c},$$

as the heavytailedness parameter tends to its limit ($\alpha \to 1$, $\sigma \to \infty$, $\tau \to 0$, $p \to 0$, $\alpha \to 1/r$ correspondingly).

Proof. Firstly let us take the Pareto distribution family. According to the results of the Remark 2.1 it follows that for any $\varepsilon \in (0, u)$ there is a constant $\alpha_0 > 1$ such that

$$\overline{B}(\varepsilon) \leq \frac{\varepsilon}{u - \varepsilon},$$

for any $\alpha \in (1, \alpha_0)$. So from the Pollaczeck-Khinchine formula (1.2), the following chain of inequalities can be taken:

$$\begin{aligned} \frac{\rho}{c} = \psi(0) \geq \psi(u) &\geq \frac{\rho}{c}\overline{F}(u) = \frac{\rho}{c}\left(1 - \int_0^{\varepsilon} \overline{B}(y)dy - \int_{\varepsilon}^{u} \overline{B}(y)dy\right) \\ &\geq \frac{\rho}{c}[1 - \varepsilon - (u - \varepsilon)\overline{B}(\varepsilon)] \geq \frac{\rho}{c}[1 - 2\varepsilon], \end{aligned}$$

for any $\alpha \in (1, \alpha_0)$, which, due to the arbitrariness of ε, gives the desired convergence. Easily we can verify that the same argument holds for the rest of the distribution families.

We see that the superheavy limit of the claim distribution in Remark 2.1 does not present a distribution and the superheavy limit of the ruin probability in Lemma 3.1 is not a decreasing function with respect to u. These deformations of the standard properties of the distribution and the ruin probability can be explained through an explosive behavior by the convergence to the limit. To preserve the standard properties in the course of the limit passage we apply a functional normalization. Namely we take a functional heavytailedness parameter, say $\alpha(u) > 1\,,\ \forall\ u \geq 0$ in the first case, such that $\alpha(u) \downarrow 1$ as $u \to \infty$.

Theorem 3.1 *If $B(x)$ belongs to the Pareto, Lognormal, Weibull, Loggamma or Burr distribution families and its heavytailedness parameter tends to its limit in the following way:*

$$\alpha(x) \downarrow 1\,,\ \ [\alpha(x) - 1]\, x \to \infty\,,$$

$$\sigma(x) \to \infty\,,\ \ \sigma(y) < 2\, y\,,\ \ \forall\, y > 0\,,$$

$$\tau(x) \downarrow 0\,,\ \ y\, \tau(y) > 1\,,\ \ \forall\, y > 0\,,$$

$$p(x) \downarrow 0\,,$$

$$\alpha(x) \downarrow 1/r\,,\ \ \frac{1}{r\, \alpha(x) - 1} = o\left(x^{1/r}\right)\,,$$

as $x \to \infty$ respectively, then their normalized tails tend to the following limits:

$$\overline{B}([\alpha(x) - 1]x) \sim \left(\frac{1}{x\, \alpha(x)}\right)^{\alpha(x)}\,,$$

$$\overline{B}\left(\exp\left\{\sigma(x)\, x - \frac{1}{2}\sigma^2(x)\right\}\right) \sim \frac{1}{\sqrt{2\,\pi}} \int_x^{\infty} \exp\left[-\frac{y^2}{2}\right]\, dy\,,$$

$$\overline{B}\left(\sqrt{\frac{\tau(x)}{2\,\pi}}\,[e\,x\,\tau(x)]^{1/\tau(x)}\right) \sim e^{-x}$$

$$\overline{B}\left(\exp\left\{\frac{x}{2^{1/p(x)}}\left(2^{1/p(x)}-1\right)\right\}-1\right) \sim \frac{1}{\Gamma[p(x)]}\int_x^\infty y^{p(x)-1}\,e^{-y}\,dy$$

$$\overline{B}([r\,\alpha(x)-1]\,(x-1)^{1/r}) \sim \left(\frac{1}{x}\right)^{\alpha(x)}$$

as $x \to \infty$. Furthermore, in the classical risk model, the corresponding ruin probabilities tend to the following limits:

$$\psi([\alpha(u)-1]\,u) \sim \frac{\rho}{c-\rho}\int_u^\infty \frac{\alpha(z)-1+z\,\alpha'(z)}{(z\,[\alpha(z)-1]\,\alpha(z\,[\alpha(z)-1]))^{\alpha(z\,[\alpha(z)-1])}}\,dz\,,$$

$$\psi\left(\exp\left\{\sigma(u)\,u-\frac{1}{2}\sigma^2(u)\right\}\right) \sim \frac{\rho}{\sqrt{2\,\pi}(c-\rho)}$$

$$\times\int_u^\infty \frac{\sigma(z)+\sigma'(z)[z-\sigma(z)]}{\exp\left\{-z\,\sigma(z)+\frac{1}{2}\sigma^2(z)\right\}}\left[\int_{\exp\left(z\,\sigma(z)-\frac{1}{2}\sigma^2(z)\right)}^\infty \exp\left(-\frac{w^2}{2}\right)\,dw\right]\,dz\,,$$

$$\psi\left(\sqrt{\frac{\tau(u)}{2\,\pi}}[e\,\tau(u)\,u]^{1/\tau(u)}\right) \sim \frac{\rho}{\sqrt{2\,\pi}(c-\rho)}\int_u^\infty e^{-\frac{\sqrt{\tau(z)}}{\sqrt{2\,\pi}}\,[e\,z\,\tau(z)]^{\frac{2}{\tau(z)}}}\,\tau'(z)$$

$$\left\{\frac{1}{2\,\sqrt{\tau(z)}}-e\,\frac{z\tau'(z)+\tau(z)}{\tau^2(z)}\,\log[e\,z\,\tau(z)]\right\}\,dz\,,$$

$$\psi\left(\exp\left\{\frac{u}{2^{1/p(u)}}\left(2^{1/p(u)}-1\right)\right\}-1\right) \sim$$

$$\frac{\rho}{c-\rho}\int_u^\infty \frac{\left(1-2^{-1/p(z)}\right)\,\exp\left\{\left(1-2^{-1/p(z)}\right)\,z\right\}}{\Gamma\left[p\left(\exp\left[\left(1-2^{-1/p(z)}\right)\,z\right]-1\right)\right]}$$

$$\times\int_{\exp\left[\left(1-2^{-1/p(z)}\right)\,z\right]-1}^\infty w^{p\left(\exp\left[\left(1-2^{-1/p(z)}\right)\,z\right]-1\right)}\,e^{-w}\,dw\,dz\,,$$

$$\psi\left([r\,\alpha(u)-1]\;(u-1)^{1/r}\right)\sim$$

$$\frac{\rho}{c-\rho}\int_u^\infty\left(\frac{1}{[r\,\alpha(z)-1]\;(z-1)^{1/r}}\right)^{\alpha\left([r\,\alpha(z)-1]\,(z-1)^{1/r}\right)}[r\,\alpha'(z)\,(z-1)^{1/r}]\,dz\,,$$

as $u\to\infty$.

Proof. Let us start with the case of Pareto claim sizes. We look for such a nomalizing function $f[x,\alpha(x)]$, that the expression $B(x\,f[x,\alpha(x)])$ remains a distribution after the passage to the limit $x\to\infty$. This means

1. $x\,f[x,\alpha(x)]\downarrow 0$, as $x\downarrow 0$,
2. $x\,f[x,\alpha(x)]\uparrow\infty$, as $x\to\infty$.

Within this framework, we find that the function

$$f[x,\alpha(x)]=\alpha(x)-1\,,$$

meets the requirements above and serves as candidate for the normalizing function.

Next we fix the value of $\alpha(u)=\alpha>1$ and we shall obtain the following uniform asymptotics for the ruin probability when $u\to\infty$,

$$\lim_{u\to\infty}\sup_{\alpha>1}\left|\frac{\psi(u\,f[u,\alpha])}{\dfrac{\rho}{c-\rho}\overline{F}(u\,f[u,\alpha])}-1\right|=\lim_{u\to\infty}\sup_{\alpha>1}\left|\frac{\psi(u\,[\alpha-1])}{\dfrac{\rho}{c-\rho}\overline{F}(u\,[\alpha-1])}-1\right|=0\,.\quad(3.1)$$

Indeed, as far $F(u\,[\alpha-1])$ represents a subexponential distribution function and $\dfrac{c-\rho}{2\,c}>0$, there exists some constant $K=K\left(\dfrac{c-\rho}{2\,c}\right)$ (see for example Embrechts et al. (1997, Lemma 1.3.5) such that for any integer $N\geq 1$ we obtain

$$\psi(u\,[\alpha-1])\quad=\quad\frac{c-\rho}{c}\sum_{n=0}^{N-1}\left(\frac{\rho}{c}\right)^n\overline{F}^{n*}(u\,[\alpha-1])$$

$$+\frac{c-\rho}{c}\sum_{n=N}^{\infty}\left(\frac{\rho}{c}\right)^n\overline{F}^{n*}(u\,[\alpha-1])\sim\quad\frac{c-\rho}{c}\sum\nolimits_{n=0}^{N-1}\left(\frac{\rho}{c}\right)^n n\,\overline{F}(u\,[\alpha-1])$$

$$+\frac{c-\rho}{c}\sum_{n=N}^{\infty}\left(\frac{\rho}{c}\right)^n\overline{F}^{n*}(u\,[\alpha-1])\leq\overline{F}(u\,[\alpha-1])$$

$$\times\left[\frac{\rho}{c-\rho}-\left(\frac{c}{c-\rho}+N-1\right)\left(\frac{\rho}{c}\right)^N+\frac{c-\rho}{c}\sum_{n=N}^{\infty}K\left(\frac{\rho}{c}\left[1+\frac{c-\rho}{2\,c}\right]\right)^n\right]$$

as $u \to \infty$. Therefore, for any real $M > 1$ we find

$$\lim_{u\to\infty} \sup_{\alpha\in[M,\infty)} \left| \frac{\psi(u\,[\alpha-1])}{\frac{\rho}{c-\rho}\overline{F}(u\,[\alpha-1])} - 1 \right| \le \left| \left(1 + N\,\frac{c-\rho}{\rho}\right)\left(\frac{\rho}{c}\right)^N \right| + \left| \frac{2\,K\,(c-\rho)}{\rho}\left(\frac{c+\rho}{2\,c}\right)^N \right| .$$

Now we take the limit for $N \to \infty$ and the limit for $M \to 1$ and we reach the asymptotic relation (3.1).

Now we see that the expression $1 - \overline{B}(x\,f[x,\alpha(x)])$ remains distribution after the passage to the limit, because and consequently:

$$\psi(u\,f[u,\alpha(u)]) \sim \frac{\rho}{c-\rho}\int_u^\infty \overline{B}(z\,f[z,\alpha(z)])\left(z\,\frac{d(f[z,\alpha(z)])}{dz} + f[z,\alpha(z)]\right) dz\,.$$

Further, we continue with the rest cases under the following normalizing functions respectively:

$$\begin{aligned}
f[x,\sigma(x)] &= \frac{1}{x}\,\exp\left\{\sigma(x)\,x - \frac{1}{2}\sigma^2(x)\right\}\,,\\
f[x,\tau(x)] &= \frac{1}{x}\sqrt{\frac{\tau(x)}{2\,\pi}}\,[e\,\tau(x)\,x]^{1/\tau(x)}\,,\\
f[x,p(x)] &= \frac{1}{x}\left[\exp\left\{\frac{x}{2^{1/p(x)}}\left(2^{1/p(x)}-1\right)\right\} - 1\right]\,,\\
f[x,\alpha(x)] &= \frac{1}{x}\,[r\,\alpha(x)-1]\,(x-1)^{1/r}\,.
\end{aligned}$$

It remains to repeat the uniform convergence and with the substitution, we reach the other ruin probability formulas.

All the five examples of heavy tailed distributions outlined in the previous statements belong to the class of subexponential distributions $\mathcal{S}$. Two members of $\mathcal{S}$ which serve as exceptions to the Lemma 3.1 are given below.

Example 3.1 *The Bektander I distribution with tail*

$$\overline{B}(x) = \left[1 + \frac{2p}{\alpha}\ln(1+x)\right]\exp\left\{-p\ln^2(1+x) - (\alpha+1)\ln(1+x)\right\}$$

with $\alpha > 0$, $p > 0$, $x > 0$.

Example 3.2 *The Bektander II distribution with tail*

$$\overline{B}(x) = \frac{1}{(1+x)^{1-p}}\exp\left\{\frac{\alpha}{p}\,[1-(1+x)^p]\right\}\,,$$

with $\alpha > 0$, $p \in [0,1]$, $x > 0$.

We consider again the classical risk model. In both cases, the normalization condition (2.1) gives

$$\alpha = 1$$

and the heavytailedness parameter tends to zero

$$p \to 0.$$

Let us denote $\psi^*(u)$ the ruin probability in the classical risk model in which the tail of claim is from the Pareto distribution: $\overline{P}(x) = 1/(1+x)^2$, for $x > 0$ (that claim distribution coincides with the Burr(2,1)).

Theorem 3.2 (Tsitsiashvili and Konstantinides (2001)) *In the classical risk model, if $B(x)$ belongs to either Bektander I or Bektander II distribution family, then for any $x > 0$, its tail converges in $\mathcal{L}^1$ to $\overline{P}(x)$:*

$$\int_0^\infty |\overline{B}(x) - \overline{P}(x)| dx \to 0,$$

and for any $u > 0$, the ruin probability tends weakly ($\Rightarrow$) to the function $\psi^*(u)$, which represents the stationary distribution tail of the waiting time in the $M/G/1/\infty$ queuing system with service time distribution $P(x)$:

$$\psi(u) \Rightarrow \psi^*(u),$$

as the heavytailedness parameter tends to zero: $p \to 0$.

Proof. Firstly let us take the example of the Bektander I distribution. Here, for any $T > 0$

$$\int_0^\infty \left|\overline{B}(x) - \overline{P}(x)\right| dx \le \int_0^T \left|\overline{B}(x) - \overline{P}(x)\right| dx + \int_T^\infty \overline{P}(x) dx + \int_T^\infty \overline{B}(x) dx$$

$$\le \; T \sup_{[0,T]} \overline{P}(x) \left|[1 + 2p\ln(1+x)] \exp[-p\ln^2(1+x)] - 1\right| + \frac{1}{1+T}$$

$$+ \; \int_T^\infty \left\{\exp[-2\ln(1+x)] + 2p\ln(1+x)\exp\left[-p\ln^2(1+x) - 2\ln(1+x)\right]\right\} dx$$

$$\le \; T \sup_{[0,T]} \left\{\left|\exp\left[-p\ln^2(1+x)\right] - 1\right| + 2p\ln(1+x)\right\} + 2\left(\frac{1}{1+T} + (1+T)e^{-T}\right).$$

Further, for any $\varepsilon > 0$ there exists a T_ε and a $p_0 > 0$ such that for any $p \in (0, p_0)$ the last expression becomes less than ε, so

$$\int_0^\infty \left|\overline{B}(x) - \overline{P}(x)\right| dx$$

$$\leq T \sup_{[0,T]} \left\{ \left| \exp\left[-p \ln^2(1+x)\right] - 1 \right| + 2p \ln(1+x) \right\} + \varepsilon \to \varepsilon,$$

as $p \longrightarrow 0$ and thus the convergence in $\mathcal{L}^1$ for the Bektander I case is obtained. Similarly for the Bektander II case, considering the uniform convergence over any finite interval $[0, T]$

$$\exp\left(\frac{1 - e^{p \ln(1+x)}}{p}\right) \to \frac{1}{1+x},$$

as $p \to 0$, the convergence of the claim tail in $\mathcal{L}^1$ is confirmed.

In both cases the convergence of the ruin probability can be verified from a well-known result of the stability theory (see for example Kalashinikov (1978) and Kalashnikov and Tsitsiashvili (1973))

Indeed, in classical risk model the convergence of $\overline{B}(x)$ to $\overline{P}(x)$ in $\mathcal{L}^1$ implies the convergence of the ruin probability $\psi(u)$ to the function $\psi^*(u)$, which represents the ruin probability with claim distribution $P(x)$ (or, in other words, it represents the stationary distribution of the waiting time in the $M/G/1/\infty$ queuing system with service time distribution $P(x)$).

Remark 3.1 For the distributions from the five examples of Theorem 3.1, the convergence in $\mathcal{L}^1$ does not hold and therefore this argument from the stability theory is not applicable.

Remark 3.2 It is possible to prove that the function $\psi^*(u)$ is continuous and so the weak convergence in Theorem 3.2 can be replaced by the point wise convergence for any $u \geq 0$.

Remark 3.3 In the superheavy tail mode, the numerics become unstable, because the values of the ruin probability become too small and the precision in calculation of the intergrals fails. This observation brings up promptly the numerical issue.

4 Light subexponential tails

Now our interest is directed on subexponential distributions that are lying in close vicinity to the light-tailed distributions. We begin with the Pareto distribution in which the parameter α is chosen as before for the role of heavytailedness parameter. Obviously, when it tends to its limit ∞ we reach the lightest distribution tail. In the second case with the Lognormal distribution, the heavytailedness parameter σ has to tend to 0 in order to find the lightest tail. Next in the example related with the Weibull case, the heavytailedness parameter τ tends to 1. Further, in the example of the Loggamma distribution, the heavytailedness parameter p tends to ∞. In the last case with the Burr distribution, the two-dimensional heavytailedness parameter (α, r) tends to (∞, ∞). For the Bektander I, distribution the heavytailedness parameter p tends to ∞. Finally, in the example with Bektander II distribution, the heavytailedness parameter p tends to 1.

We proceed to the limit distributions.

Lemma 4.1 *If $B(x)$ belongs to one of the distribution families: Pareto, Lognormal, Weibull, Loggamma, Burr or Bektander II then its tail tends to a limit distribution:*

$$\overline{B}(x) \to \overline{D}(x), \tag{4.1}$$

as the corresponding heavytailedness parameter reaches its limit ($\alpha \to \infty$, $\sigma \to 0$, $\tau \to 1$, $p \to \infty$, $(\alpha, r) \to (\infty, \infty)$ and $p \to 1$ respectively). Namely in the Pareto, Lognormal, Loggamma and Burr cases, the limit distribution $\overline{D}(x)$ represents a step function:

$$\overline{D}(x) = \begin{cases} 1, & 0 \le x < 1, \\ C, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$$

for some constant $C \in (0,1)$, in the Weibull and Bektander II cases it represents an exponential distribution:

$$\overline{D}(x) = e^{-x}\,.$$

Furthermore in the Pareto, Lognormal, Weibull, Loggamma, Burr and Bektander II cases, the $\mathcal{L}^1$ convergence holds:

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy \to 0.$$

Proof. For the Pareto distribution family it is easy to find:

$$\overline{B}(x) = \begin{cases} 1, & 0 \le x \le 1 - \frac{1}{\alpha}, \\ \frac{1}{x^\alpha}(1 - \frac{1}{\alpha})^\alpha & 1 - \frac{1}{\alpha} < x < \infty, \end{cases} \to \begin{cases} 1, & 0 \le x < 1, \\ e^{-1}, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$$

as $\alpha \to \infty$. Thus (4.1) holds with $C = e^{-1}$.

For the Lognormal distribution family for any $x > 0$ holds:

$$\overline{B}(x) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln x}{\sigma} + \frac{\sigma}{2}}^\infty \exp\left\{-\frac{y^2}{2}\right\} dy \to \begin{cases} 1, & 0 \le x < 1, \\ 1/2, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$$

as $\sigma \to 0$. Here the relation (4.1) holds with $C = 1/2$.

For the Weibull distribution family the following limit distribution can be found

$$\overline{B}(x) \longrightarrow e^{-x},$$

as $\tau \longrightarrow 1$, because $[\Gamma(1+1/\tau)]^{\tau} \longrightarrow 1$.

At the Loggamma distribution family for any $\varepsilon \in (0,x)$,

$$\overline{B}(x) = \frac{1}{\Gamma(p)} \int_{A_p(x)}^{\infty} w^{p-1} e^{-w} dw,$$

where

$$A_p(x) = \frac{2^{1/p}}{2^{1/p}-1} \ln(x+1) \sim p \frac{\ln(x+1)}{\ln 2},$$

as $p \to \infty$. Let us take

$$z = \frac{\ln(x+1)}{\ln 2}.$$

For $x < 1$, as $z < 1$ the asymptote can be found with the help of the Stirling formula

$$\begin{aligned} \overline{B}(x) &\sim \frac{1}{\Gamma(p)} \int_{pz}^{\infty} w^{p-1} e^{-w} dw \\ &\geq 1 - \frac{1}{\Gamma(p)} (pz)^{p-1} e^{-pz} pz \sim 1 - \frac{p\left(ze^{1-z}\right)^p}{\sqrt{2\pi(p-1)}} \to 1, \end{aligned} \tag{4.2}$$

as $p \to \infty$, because the function $w^{p-1}e^{-w}$ reaches its maximum at $w = p-1 \sim p$ and in turn the function ze^{1-z} reaches its maximum equal to 1 at $z = 1$. Furthermore, this convergence is uniform with respect to $x \in [0, 1-\varepsilon]$ for any $\varepsilon \in (0,1)$.

For $x > 1$, as $z > 1$ the asymptote can be found similarly

$$\begin{aligned} \overline{B}(x) &\sim \frac{1}{\Gamma(p)} \int_{pz}^{2p} w^{p-1} e^{-w} dw + \frac{1}{\Gamma(p)} \int_{2p}^{\infty} w^{p-1} e^{-w} dw \\ &\leq \frac{1}{\Gamma(p)} p^p e^{-pz} + \frac{2}{\Gamma(p)} \int_{p}^{\infty} [2(p-1)]^{p-1} e^{-p+1} e^{-u} du \\ &\sim \frac{p\left(ze^{1-z}\right)^p}{z\sqrt{2\pi(p-1)}} + \frac{1}{(e/2)^p \sqrt{2\pi(p-1)}} \to 0, \end{aligned}$$

as $p \to \infty$, with $u = w/2$, because the function $w^{p-1}e^{-w/2}$ reaches its maximum at $w = 2(p-1)$.

For $x = 1$ let us take

$$\overline{B}(1) \sim \frac{1}{\Gamma(p)} \int_{p}^{\infty} w^{p-1} e^{-w} dw \longrightarrow C_g,$$

as $p \to \infty$.

For the Burr distribution family for fixed r, it follows from (2.2) and Stirling's formula

$$\kappa \sim \alpha \left[\frac{r}{\Gamma(1/r)}\right]^r,$$

as $\alpha \to \infty$, from where

$$\overline{B}(x) = \left[1 + \frac{x^r}{\kappa}\right]^{-\alpha} \sim \left(1 + \frac{x^r}{\alpha}\left[\frac{1}{r}\,\Gamma\left(\frac{1}{r}\right)\right]^r\right)^{-\alpha} \to \exp\left\{-\left[\frac{1}{r}\,\Gamma\left(\frac{1}{r}\right)\right]^r x^r\right\},$$

as $\alpha \to \infty$. But

$$\left[\frac{1}{\tau}\,\Gamma\left(\frac{1}{\tau}\right)\right]^\tau x^\tau \to \begin{cases} 0, & 0 \le x < 1, \\ e^{-\gamma}, & x = 1, \\ \infty, & 1 < x < \infty, \end{cases}$$

with

$$\left[\frac{1}{\tau}\,\Gamma\left(\frac{1}{\tau}\right)\right]^\tau \sim \left[1 + \frac{1}{\tau}\,\Gamma'(1) + o\left(\frac{1}{\tau}\right)\right]^\tau \to e^{-\gamma},$$

as $\tau \to \infty$, where $\gamma = 0.5772156649$ the Euler's constant, from where (4.1) follows with $C = \exp(-e^{-\gamma})$

$$\lim_{\tau\to\infty}\lim_{\alpha\to\infty} \overline{B}(x)$$

$$= \lim_{\tau\to\infty} \exp\left\{-\left[\frac{1}{\tau}\,\Gamma\left(\frac{1}{\tau}\right)\right]^\tau x^\tau\right\} = \begin{cases} 1, & 0 \le x < 1, \\ \exp(-e^{-\gamma}), & x = 1, \\ 0, & 1 < x < \infty, \end{cases} \tag{4.3}$$

and this convergence is uniform on $x \in [0, 1-\varepsilon]$ for any $\varepsilon \in (0,1)$.

Next we examine the Bektander II distribution. Again from the substitution $\alpha = 1$ it follows the limit

$$\overline{B}(x) = \frac{1}{(1+x)^{1-p}} \exp\left\{\frac{1-(1+x)^p}{p}\right\} \longrightarrow e^{-x},$$

as $p \to 1$.

Now, for the convergence in $\mathcal{L}^1$, let us note that for any $\varepsilon > 0$ and all claim distributions B

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy = 2\int_0^1 \left[1 - \overline{B}(y)\right] dy \le 2\int_0^{1-\varepsilon} \left[1 - \overline{B}(y)\right] dy + 2\varepsilon\,. \tag{4.4}$$

This relation gives in the Pareto case

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy = 2\int_{(\alpha-1)/\alpha}^1 dy = \frac{2}{\alpha} \to 0,$$

as $\alpha \to \infty$.

In the Lognormal case, for any $\varepsilon > 0$ if we choose $\sigma_\varepsilon > 0$ such that

$$\int_0^{1-\varepsilon} \int_{-\infty}^{\frac{\ln y}{\sigma_\varepsilon}+\frac{\sigma_\varepsilon}{2}} \exp\left\{-\frac{u^2}{2}\right\} \frac{du}{\sqrt{2\pi}} dy < \varepsilon,$$

then for every $\sigma \in (0, \sigma_\varepsilon)$

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy \le 2\int_0^{1-\varepsilon} \int_{-\infty}^{\frac{\ln y}{\sigma}+\frac{\sigma}{2}} \exp\left\{-\frac{u^2}{2}\right\} \frac{du}{\sqrt{2\pi}} dy + 2\varepsilon \le 4\varepsilon,$$

which gives the convergence in $\mathcal{L}^1$ as $\sigma \to 0$.

We take the Weibull distribution. Let us see that for any $T > 1$, $\tau > 1/2$

$$\begin{aligned}
\int_0^\infty \left|\overline{B}(y) - e^{-y}\right| dy &= \int_0^\infty e^{-y} \left|\exp\left\{-\left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^\tau + y\right\} - 1\right| dy \\
\le & \left(\int_0^T + \int_T^\infty\right) \left|\exp\left\{-\left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^\tau + y\right\} - 1\right| dy \\
\le & \; T \sup_{[0,T]} \left|\exp\left\{-\left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^\tau + y\right\} - 1\right| + e^{-T} \\
+ & \int_T^\infty \exp\left\{-\inf_{\tau\in[1/2,1]} \left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^{1/2}\right\} dy\,.
\end{aligned}$$

But $\forall \varepsilon > 0$ there exists T_ε such that

$$e^{-T_\varepsilon} + \int_{T_\varepsilon}^\infty \exp\left\{-\inf_{\tau\in[1/2,1]} \left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^{1/2}\right\} dy < \frac{\varepsilon}{2}\,.$$

As far as $y^\tau \to y$, uniformly on $[0, T_\varepsilon]$ and

$$\left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau \to 1\,,$$

as $\tau \to 1$, we can choose $\tau_\varepsilon \in (1/2, 1)$ such that $\forall \tau \in (\tau_\varepsilon, 1)$

$$\sup_{[0,T_\varepsilon]} \left|\exp\left\{-\left[\Gamma\left(1+\frac{1}{\tau}\right)\right]^\tau y^\tau + y\right\} - 1\right| < \frac{\varepsilon}{2T_\varepsilon}\,,$$

and therefore

$$\int_0^\infty \left|\overline{B}(y) - e^{-y}\right| dy < \varepsilon\,.$$

In the Loggamma case, it follows from the relation (4.4)

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy \le 2\int_0^{1-\varepsilon} \left[1 - \frac{1}{\Gamma(p)}\int_{A_p(y)}^\infty w^{p-1}e^{-w}dw\right] dy + 2\varepsilon \le 4\varepsilon,$$

as $p \to \infty$. In the last inequality we used the uniform convergence on $[0, 1-\varepsilon]$ in (4.2).

Similarly, in the Burr distribution

$$\int_0^\infty \left|\overline{B}(y) - \overline{D}(y)\right| dy \le 2\int_0^{1-\varepsilon} \left[1 - \overline{B}(y)\right] dy + 2\varepsilon \le 4\varepsilon,$$

as $(\alpha, r) \to (\infty, \infty)$ or $\alpha \to \infty$, because of the uniform convergence on $[0, 1-\varepsilon]$ in (4.3).

We examine the Bektander II distribution. Let us notice that for any $T > 0$ and $p > 1/2$

$$\int_0^\infty \left|\overline{B}(y) - e^{-y}\right| dy \le \int_T^\infty \left|e^{-y}\right| dy + \int_T^\infty \left|\overline{B}(y)\right| dy + T \sup_{0\le y\le T}\left|\overline{B}(y) - e^{-y}\right|$$

$$\le e^{-T} + \exp\left\{\frac{1-(1+T)^p}{p}\right\}$$

$$+T\left[e^2\left(1 - \frac{1}{(1+T)^{1-p}}\right) + \sup_{0\le y\le T}\left[-1 + \exp\left\{\frac{1-(1+y)^p + py}{p}\right\}\right]\right]$$

$$\le e^{-T} + \exp\left\{1 - (1+T)^{1/2}\right\}$$

$$+T\left[e^2\left(1 - \frac{1}{(1+T)^{1-p}}\right) - 1 + \exp\left\{2\left[1-(1+T)^p + pT\right]\right\}\right],$$

where for any $\varepsilon > 0$, T_ε can be chosen such that the first two terms in the last sum are less than $\varepsilon/2$ and there exists a $p_\varepsilon > 1/2$ such that the last term in the sum become less than ε for any $p \in (p_\varepsilon, 1)$. Thus

$$\int_0^\infty \left|\overline{B}(y) - e^{-y}\right| dy \le 2\varepsilon,$$

and the convergence in $\mathcal{L}^1$ is proved.

The stability theory with respect to $G|G|1|\infty$ systems, renders the following picture: If we have as input characteristics the distribution function of the service times (claims distribution function for our risk model) and as output characteristics the stationary distribution of the waiting times (ruin probability for our risk model), the stability means that the convergence in $\mathcal{L}^1$ of the input characteristics implies the weak convergence of the output characteristics. If in our cases the stationary distribution of waiting time is continuous, then the weak convergence is equivalent to point convergence.

Theorem 4.1 (Kalashnikov (1978)) *In the classical risk model, if the claim size distribution $B(x)$ belongs to one of the following distribution families: Pareto, Lognormal, Loggamma, Burr, then the ruin probability tends to limit waiting time distribution tail in the $M/D/1$ queuing model (see Prabhu (1998, Th. 2.17), Asmussen (2000, Cor. III.3.6) and Erlang (1909))*

$$\psi(u) \to 1 - \left(1 - \frac{\rho}{c}\right) \sum_{n=0}^{[u]} \frac{1}{n!} \left[-\frac{\rho}{c}(u-n)\right]^n e^{-\frac{\rho}{c}(u-n)}, \qquad 0 \le u < \infty,$$

as the corresponding parameter of heaviness reaches its limit ($\alpha \to \infty$, $\sigma \to 0$, $p \to \infty$ and $(\alpha, r) \to (\infty, \infty)$ respectively).

If $B(x)$ belongs to Weibull or Bektander II distribution family, the ruin probability tends to the $M/M/1$ waiting time distribution:

$$\psi(u) \longrightarrow \frac{\rho}{c} \exp\left\{-\left(1 - \frac{\rho}{c}\right) u\right\},$$

as $\tau \to 1$ and $p \to 1$ respectively.

Proof. The proof in the first case comes from a result in the stability theory of the Lindley chain as appears in Kalashnikov (1978, Th.V.5.5) (see also Kalashnikov and Tsitsiashvili (1973, Th.2) or Kalashnikov (2000, Th.1)). Here the classical risk model is described as queuing system $G/G/1/\infty$. The conditions of this theorem are satisfied through the appropriate choice of the test function in this chain.

For Weibull or Bektander II distributions, the ruin probability convergence follows again from the result of the stability theory Kalashnikov (1978, Th.5.3.1) (see also Kalashnikov and Tsitsiashvili (1973, Th.2), Kalashnikov (2000, Th.1) or Kalashnikov (2001)). The limit of the ruin probability corresponds to the waiting time distribution in $M/M/1$ queuing system (see for example Prabhu (1998, Th.1.15)).

Remark 4.1 We see that if $B(x)$ belongs to the Bektander I distribution family, then its tail tends to a limit equal to zero:

$$\overline{B}(x) \to 0,$$

as $p \to \infty$.

Namely, let us make the substitution $\alpha = 1$. Then we take

$$\overline{B}(x) = \frac{1 + 2p\ln(1+x)}{(1+x)^2}\exp[-p\ln^2(1+x)] \longrightarrow 0,$$

as $p \to \infty$.

Hence in classical risk model, if the claim size distribution $B(x)$ belongs to one of the Bektander I distribution family, the ruin probability tends to $\frac{\rho}{c}$:

$$\psi(u) \to \frac{\rho}{c},$$

as $p \to \infty$.

Indeed, for the Bektander I distribution we see that for any $\varepsilon \in (0,u)$ there is a constant $p_0 > 1$ such that

$$\overline{B}(\varepsilon) \leq \frac{\varepsilon}{u-\varepsilon},$$

for any $p > p_0$. So, from Pollaczeck-Khinchine formula (1.2) the following chain of inequalities can be taken:

$$\begin{aligned} \frac{\rho}{c} &= \psi(0) \geq \psi(u) \geq \frac{\rho}{c}\overline{F}(u) = \frac{\rho}{c}\left(1 - \int_0^\varepsilon \overline{B}(y)dy - \int_\varepsilon^u \overline{B}(y)dy\right) \\ &\geq \frac{\rho}{c}[1 - \varepsilon - (u-\varepsilon)\overline{B}(\varepsilon)] \geq \frac{\rho}{c}[1-2\varepsilon], \end{aligned}$$

for any $p > p_0$.

We see in this remark that the lighter limit of the Bektander I claim distribution does not represent a distribution and the lighter limit of the ruin probability is not a decreasing function with respect to u. These deformations of the standard properties of the distribution function and the ruin probability expresses a tail explosion through the convergence to the limit. As we have done in the previous section we proceed to a functional normalization. Namely we take a functional heavytailedness parameter $p(u) > 1\,,\ \ \forall u \geq 0$, such that $p(u) \to \infty$ as $u \to \infty$.

Theorem 4.2 *If $B(x)$ belongs to the Bektander I distribution family with its heavytailedness parameter $p(x) \to \infty$, then the normalized tail tend to the following limit:*

$$\overline{B}\left(\exp\left\{\sqrt{\frac{x}{p(x)}}\right\} - 1\right) \sim (1 + 2\sqrt{x\,p(x)})\exp\left\{-x - 2\sqrt{\frac{x}{p(x)}}\right\},$$

as $x \to \infty$. Further, in the classical risk model, the ruin probability tends to the following limit:

$$\psi\left(\exp\left\{\sqrt{\frac{u}{p(u)}}\right\}-1\right)\sim$$

$$\frac{\rho}{c-\rho}\int_u^\infty \frac{\sqrt{z\,p(z)}+2\,z\,p(z)}{2\,p(z)}\left(\frac{1}{z}-\frac{p'(z)}{p(z)}\right)\exp\left\{-z-\sqrt{\frac{z}{p(z)}}\right\}dz\,.$$

as $u\to\infty$.

Proof. Indeed, again from the relation (1.2) and the property of the subexponentiality we find

$$\psi(u)\sim\frac{\rho}{c-\rho}\overline{F}(u)=\frac{\rho}{c-\rho}\int_u^\infty \overline{B}(y)dy\,.$$

Now we look for a normalizing function in the form $f(x,p(x))$, for which the expression $1-\overline{B}(x\,f(x,p(x)))$ remains distribution after the passage to the limit:

$$\begin{aligned}\psi(u\,f(u,p(u))) &\sim \frac{\rho}{c-\rho}\int_{u\,f(u,p(u))}^\infty \overline{B}(y)\,dy\\ &\sim \frac{\rho}{c-\rho}\int_u^\infty \overline{B}(z\,f(z,p(z)))\left(\frac{d[f(z,p(z))]}{dz}z+f(z,p(z))\right)dz\,.\end{aligned}$$

Namely, we find the following normalizing function:

$$f(x,p(x))=\frac{1}{x}\left(\exp\left\{\sqrt{\frac{x}{p(x)}}\right\}-1\right).$$

And after the substitution we reach the ruin probability

$$\psi\left(\exp\left\{\sqrt{\frac{u}{p(u)}}\right\}-1\right)\sim\frac{\rho}{c-\rho}\int_u^\infty(1+2\sqrt{z\,p(z)})$$

$$\times\exp\left\{-z-2\sqrt{\frac{z}{p(z)}}\right\}\frac{1}{2\sqrt{p(z)}}\left(\frac{1}{\sqrt{z}}-\frac{p'(z)\sqrt{z}}{p(z)}\right)\exp\left\{\sqrt{\frac{z}{p(z)}}\right\}dz\,.$$

from where we find our result.

Finally, we gather together the previous description of the limit behavior of the subexponential distribution families in the Table 1.

Table 1 *Limit claim distribution tails and ruin probabilities.*

Distribution	$\lim \overline{B}(x)$	$\lim \psi(u)$
Pareto	$\overline{D}(x) = \begin{cases} 1, & 0 \leq x < 1, \\ e^{-1}, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$	$1 - \left(1 - \frac{\rho}{c}\right) \cdot \sum_{n=0}^{[u]} \frac{1}{n!} \left[-\frac{\rho}{c}(u-n)\right]^n e^{-\frac{\rho}{c}(u-n)}$
Lognormal	$\overline{D}(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 1/2, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$	$1 - \left(1 - \frac{\rho}{c}\right) \cdot \sum_{n=0}^{[u]} \frac{1}{n!} \left[-\frac{\rho}{c}(u-n)\right]^n e^{-\frac{\rho}{c}(u-n)}$
Weibull	e^{-x}	$\rho/c \exp\{-(1-\rho/c)\,u\}$
Loggamma	$\overline{D}(x) = \begin{cases} 1, & 0 \leq x < 1, \\ C_g, & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$	$1 - \left(1 - \frac{\rho}{c}\right) \cdot \sum_{n=0}^{[u]} \frac{1}{n!} \left[-\frac{\rho}{c}(u-n)\right]^n e^{-\frac{\rho}{c}(u-n)}$
Burr	$\overline{D}(x) = \begin{cases} 1, & 0 \leq x < 1, \\ \exp\left(-e^{-\gamma}\right), & x = 1, \\ 0, & 1 < x < \infty, \end{cases}$	$1 - \left(1 - \frac{\rho}{c}\right) \cdot \sum_{n=0}^{[u]} \frac{1}{n!} \left[-\frac{\rho}{c}(u-n)\right]^n e^{-\frac{\rho}{c}(u-n)}$
Benktander II	e^{-x}	$\rho/c \exp\{-(1-\rho/c)\,u\}$

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