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The limiting copula of the two largest order statistics of independent and identically distributed samples

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Abstract: In this paper we obtain the expression of the *bi-extremal* copula, the copula pertaining to the limit distribution of the two largest order statistics of iid samples. Properties of the bi-extremal copula are derived, including the Kendall's τ coefficient and the tail dependence coefficient. Theoretical and simulated examples are worked out. As applications we propose the construction of *meta-bi-extremal* distributions and provide an illustration of a modeling situation focusing in extreme risks in the field of finance.

Key words: Catastrophic risks, copulas, meta-extremal distributions, r-largest extreme value model.

1 Motivation

Lately, media have been reporting record breaks more frequently. Experts' discussions on the related catastrophes typically include arguments based on the ozone and ultraviolet radiation levels, on the *El Niño* effect, on the global warming effect, or on the globalization effect.

Recent examples of the catastrophic consequences of extreme events are the flooding in New Orleans in 2005, the occurrence tsunamis in the Indian ocean in 2004, or a stock market crash in 2002, among others. In all of these events, not just the largest observation was extreme, but also the second largest, or more generally the r-largest for some $r \geq 2$. Given n realizations of a stationary process $\{X_t\}_{t\geq 1}$, worst possible scenarios are characterized by their r-largest order statistics being all extreme. This situation calls for statistical models suitable for computing accurate probabilities of new records in a near future. The copula associated to the distribution of the r-largest statistics contains all information about this type of dependence structure.

2 Preliminaries

Copulas

Let (X_1, X_2) be a random variable (rv) in $\Re^2$ with joint distribution function (cdf) F and margins F_i, $i = 1, 2$. Suppose that the margins X_1 and X_2 are respectively transformed into Uniform$(0, 1)$ rv's U and V. If F_1 and F_2 are continuous, this transformation may be obtained through the probability integral transformation $(x_1, x_2) \mapsto (F_1(x_1), F_2(x_2))$.

The joint distribution function $C(\cdot)$ of $(F_1(X_1), F_2(X_2))$ is the copula of the random vector (X_1, X_2), or equivalently, the copula pertaining to F. It follows that

$$F(x_1, x_2) = C(F_1(x_1), F_2(x_2)) \; . \tag{2.1}$$

If F_1 and F_2 are continuous, then C is unique. Conversely, if C is a copula and F_1 and F_2 are cdf's, the function F defined in (2.1) is a joint cdf with margins F_i, $i = 1, 2$, (Sklar, 1959). If we assume that each F_i and C are differentiable, then the joint density $f(x_1, x_2)$ of (X_1, X_2) can be written as

$$f(x_1, x_2) = c(F_1(x_1), F_2(x_2)) \cdot \prod_{i=1}^{2} f_i(x_i) \; , \tag{2.2}$$

where $c(u, v) = \frac{\partial C^2(u,v)}{\partial u \partial v}$ is the copula density, and f_i is the density function of X_i, $i = 1, 2$. In (2.2) we note the decomposition of the joint density in two parts. One describes the dependence structure, and the other describes the marginal behavior of each component.

The r-largest model

Let X_i, $i = 1, 2, ...n$, be independent and identically distributed (iid) random variables with common density function f and cdf F, and let $M_{k,n}$ represent the kth largest statistic of $\{X_1, ..., X_n\}$. The density $f_{k,n}$ and cdf $F_{k,n}$ of $M_{k,n}$ are well known and given by

$$f_{k,n}(z) = \frac{n!}{(n-k)!(k-1)!}[F(z)]^{n-k}[1 - F(z)]^{k-1} f(z) \tag{2.3}$$

and

$$F_{k,n}(z) = \sum_{j=n-k+1}^{n} \binom{n}{j} [F(z)]^j [1 - F(z)]^{n-j} \tag{2.4}$$

for z in the support of F.

However, the above expressions are not very useful since in most practical situations F is unknown. The extreme value theory provides an asymptotic result, the generalization of the Fisher and Tippett theorem (Fisher and Tippett, 1928), which is independent of the parent distribution F, see Theorem 4.2.3 of Embrechts, Klüppelberg and Mikosch (1997). Let $G_1(\xi, \mu, \sigma)$ represent the family of the generalized extreme value distributions (GEV), where $\xi \in \Re$, $\mu \in \Re$, $\sigma > 0$. If there exist sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that $Pr\{(M_{1,n} - b_n)/a_n \leq z\} \to G_1(z)$ as $n \to \infty$, for some non-degenerate distribution function G_1, then G_1 is of the type of one of the three types of extreme value distributions (types I, II, II, respectively, the Gumbel, the Frechét, and the Weibull distributions), and, for fixed k,

$$Pr\{(M_{k,n} - b_n)/a_n \leq z\} \;\to\; G_k(z)$$

on the set of z for which $1 + \xi(z - \mu)/\sigma > 0$, where

$$G_k(z) = G_1(z) \sum_{s=0}^{k-1} \frac{[-\log(G_1(z))]^s}{s!} \ . \tag{2.5}$$

By taking derivatives of (2.5) we obtain the densities

$$g_k(z) = \exp\{-\Upsilon(z)\}(-\Upsilon'(z)) \left(\sum_{s=0}^{k-1} \frac{(\Upsilon(z))^s}{s!} \right) + \exp\{-\Upsilon(z)\} \left(\sum_{s=0}^{k-1} \frac{s(\Upsilon(z))^{s-1}}{s!} \right) \ , \tag{2.6}$$

where $\Upsilon(z) = -\log(G_1(z))$, $\Upsilon'(z) = \partial \Upsilon(z)/\partial z$.

Suppose a set of observations of the r-largest order statistics is available. Again, the exact joint distribution of $(M_{1,n}, M_{2,n}, \cdots, M_{r,n})$ cannot be used for inferences since it depends on F. With the appropriate re-scaling of the random vector, a limit joint distribution may be obtained (see Weissman, 1978; Smith, 1986; Tawn, 1988). If F belongs to the maximum domain of attraction of some extreme value distribution $G_1(\xi, \mu, \sigma)$ with sequences of normalizing constants $\{a_n > 0\}$ and $\{b_n\}$, then the limiting distribution as $n \to \infty$ of

$$\left(\frac{M_{1,n} - b_n}{a_n}, ..., \frac{M_{r,n} - b_n}{a_n} \right) \tag{2.7}$$

is a member of a family having the following joint density function (Smith, 1986)

$$g(z_1, ..., z_r) = \exp\left\{ -\left[1 + \xi \left(\frac{z_r - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi}} \right\} \times \prod_{k=1}^{r} \frac{1}{\sigma} \left[1 + \xi \left(\frac{z_k - \mu}{\sigma} \right) \right]^{-\frac{1}{\xi} - 1} \ , \tag{2.8}$$

on the set of z_k, $k = 1, ..., r$, for which $1 + \xi(z_k - \mu)/\sigma > 0$, and $z_r \leq z_{r-1} \leq \cdots \leq z_1$. In (2.8) the unknown scaling constants are absorbed into the location and scale parameters (μ and σ) since these equations provide the expression for the limiting distribution up to type. The case $r = 1$ reduces to the GEV family of distributions. For a proof of (2.8) based on a point process limit representation of the extreme value behavior of the underlying process, see Coles (2001).

The case $\xi = 0$ is given in Weissman (1978). It is interpreted as the limiting form of (2.8) as $\xi \to 0$,

$$g(z_1, ..., z_r) = \exp\left\{ -\exp\left\{ -\left(\frac{z_r - \mu}{\sigma} \right) \right\} \right\} \times \prod_{k=1}^{r} \frac{1}{\sigma} \exp\left\{ -\left(\frac{z_k - \mu}{\sigma} \right) \right\} \ , \tag{2.9}$$

where $\mu \in \Re$, $\sigma > 0$, and $z_r \leq z_{r-1} \leq \cdots \leq z_1$. In (2.9), the case $r = 1$ reduces to the Gumbel family of densities. Figure 1 illustrates and shows the limit densities and cdf's of the 5 largest statistics and for $\mu = 0.0$, $\sigma = 1$, and $\xi = 0.2$.

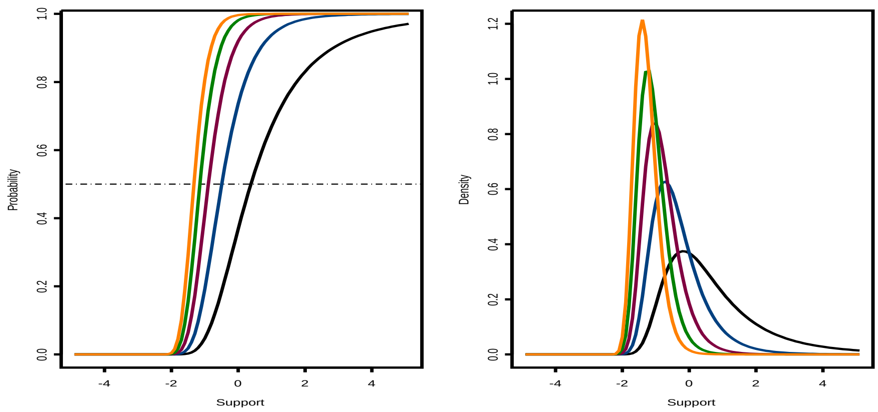

Figure 1 *At the left hand side the cdf's and at the right hand side the densities of the* 5 *largest statistics and for* $\mu = 0.0$*,* $\sigma = 1$*, and* $\xi = 0.2$*.*

There are many situations where a multivariate data set shows a strong, positive association, resembling the dependence structure pertaining the r-largest model (2.8), including the particular features of the support of the multivariate distribution. Examples may be found in flooding or extreme sea levels occurring in nearby regions, where the focus is usually on estimation of risks associated to some small probability scenarios. However, model (2.5) may not provide a good fit for (some of) the components. In such situations one would like to be free to find the best possible marginal fits, which could then be linked by means of the copula pertaining to the r-largest model. This construction is supported by the Sklar theorem (Sklar, 1959) and it was the motivation for this paper.

Accordingly, in this paper we derive the expression for the copula C_{BIX}, the *bi-extremal* copula pertaining to the asymptotic distribution of the 2-largest order statistics from a stationary process. We refer the reader to Leadbetter et al. (1983) for the development of theoretical conditions for the extreme value models to hold in the case X_i, $i = 1, 2, ...$ is a stationary series. This copula is non-parametric and depends on F only through the assumptions in the limit theorems.

Two related works are Schmitz (2004) and Avérous et al. (2005). Schmitz (2004) obtains the dependence structure of the minimum and maximum of n iid random variables by determining their copula. Avérous et al. (2005) compare the degree of association presented by any two pairs of order statistics from the same continuous distribution. To this end they use the concept of bivariate monotone regression dependence. It is also shown that the copula associated with a pair of order statistics does not depend on the parent distribution F, although an expression for the copula is not worked out. Another related work is Anjos et al. (2005), where it is given a copula representation of the joint distribution function of r-th and s-th order statistics corresponding to a pair of random variables.

The remaining of this paper is organized as follows. In Section 3 we derive the expression for the bi-extremal copula. We explore this dependence structure by computing the expressions for the Kendall's tau, the Spearman's ρ, and the tail dependence coefficient. Relations between the Kendall's tau and the Spearman's ρ coefficients are obtained and linked to previous important works. We work out examples and simulations, propose an algorithm for simulating from the bi-extremal copula, and suggest constructing the class of meta-bi-extremal distributions. The resulting distributions are suitable for modeling some extreme situations associated with catastrophes. We provide an illustration using data from the field of finance. In Section 4 we discuss some further issues and comment on extensions of this work.

3 The bi-extremal copula

We consider the special case $r = 2$. Without loss of generality let $\mu = 0$ and $\sigma = 1$. The limit marginal distributions of the maximum $M_{1,n}$ and the second largest $M_{2,n}$ are given by

$$G_1(z) = \exp\{-\Upsilon(z)\} \text{ , and } G_2(z) = \exp\{-\Upsilon(z)\}[1 + \Upsilon(z)] \ , \tag{3.1}$$

where, as previously defined, $\Upsilon(z) = -\log(G_1(z))$. The densities are, respectively,

$$g_1(z) = -\exp\{-\Upsilon(z)\}\Upsilon'(z) \ , \tag{3.2}$$

and

$$g_2(z) = -\exp\{-\Upsilon(z)\}\Upsilon'(z)\Upsilon(z) \ . \tag{3.3}$$

In the bivariate case the asymptotic joint density is given by

$$g(z_1, z_2) = \exp\{-\Upsilon(z_2)\}\, \Upsilon'(z_1)\Upsilon'(z_2) \ , \tag{3.4}$$

where $z_2 \le z_1$, and $z_k : 1 + \xi z_k > 0$ in the case $\xi \ne 0$, and $z_k \in \Re$, when $\xi = 0$, $k = 1, 2$.

We now obtain the density c_{BIX} of the bi-extremal copula.

Theorem 3.1. *(The bi-extremal copula density function). The density function of the bi-extremal copula pertaining to the* 2*-largest model with limit density function (3.4) is given by:*

$$c_{BIX}(u, v) = \frac{\psi(v)}{u(v - \psi(v))}, \tag{3.5}$$

where $0 \le u, v, \psi(v) \le 1$*,* $\psi(v) \le u$*,* $\psi(v) < v$*, and* $\psi(v)$ *is implicitly defined as* $v = \psi(v)(1 - \log(\psi(v)))$*.*

Proof. By plugging the expressions (3.2), (3.3), and (3.4) in (2.2), we obtain $c_{BIX}(u,v) = 1/(u\Upsilon(z_2))$. Let $\psi(v) = \exp\{-\Upsilon(z_2)\} = G_1(z_2)$. Then, $\Upsilon(z_2) = -\log(\psi(v))$, and $c_{BIX}(u,v) = (-1)/(u\log(\psi(v)))$. Note that $z_2 \leq z_1 \Rightarrow \psi(v) \leq u$, and from (3.1) and since $G_1(z_2) < G_2(z_2)$ we have $\psi(v) < v$. Since $v = \exp\{-\Upsilon(z_2)\}(1 + \Upsilon(z_2))$, we have $v = \psi(v)(1 - \log(\psi(v)))$ and the result follows□

Remark 3.1. By definition, the function ψ is the inverse of the function $x \mapsto x(1 - \log(x))$. As noted by a referee, for the numerical evaluation of the function ψ, which we will need later for simulation, one should note that it can be expressed with the help of the so called Lamberts W-function, which is implemented e.g. in MAPLE.

Remark 3.2. From *Proposition 10* in Avérous et al. (2005), we have that the bi-extremal copula is also the copula of $(-M_{n,n}, -M_{n-1,n})$, (minus the smallest and second smallest order statistics) associated to a random sample of size n of some continuous distribution.

It is easy to show that the limit bivariate cdf is given by

$$G(z_1, z_2) = \begin{cases} \exp\{-\Upsilon(z_2)\}\left[1 + \Upsilon(z_2) - \Upsilon(z_1)\right] & \text{if } \ z_2 \leq z_1 \\ \exp\{-\Upsilon(z_1)\} & \text{if } \ z_2 > z_1. \end{cases} \tag{3.6}$$

From the Sklar theorem we have

$$C_{BIX}(u,v) = G(G_1^{-1}(u), G_2^{-1}(v)), \tag{3.7}$$

where $G_i^{-1}(\cdot)$ represents the inverse function of G_i, $i = 1, 2$.

To obtain the inverse functions $G_i^{-1}(\cdot)$, $i = 1, 2$, we note that $G_1(z_1) = u = \exp\{-\Upsilon(z_1)\}$ implies $\Upsilon(z_1) = -\log(u)$, and then the inverse of G_1 is $z_1 = ((-\log(u))^{-\xi} - 1)/\xi$. Now, $G_2(z_2) = \exp\{-\Upsilon(z_2)\}(1 + \Upsilon(z_2)) = v$, and since by definition $v = \psi(v)(1 - \log(\psi(v)))$, $\psi(v) = G_1(z_2) \Rightarrow -\log(\psi(v)) = \Upsilon(z_2)$. The inverse of G_2, $z_2 = \frac{(\frac{v-\psi(v)}{\psi(v)})^{-\xi} - 1}{\xi}$, follows from $v = \psi(v)(1 + \Upsilon(z_2)) \Rightarrow \frac{v-\psi(v)}{\psi(v)} = (1 + \xi z_2)^{-1/\xi}$. Note that $\Upsilon(z_2) = (1 + \xi\frac{(\frac{v-\psi(v)}{\psi(v)})^{-\xi} - 1}{\xi})^{-1/\xi} = \frac{v-\psi(v)}{\psi(v)}$. We are now in position for obtaining the copula C_{BIX}.

Theorem 3.2. *(The copula C_{BIX}). The bi-extremal copula C_{BIX} pertaining to the limit distribution of the two largest order statistics from a sequence of iid random variables is given by*

$$C_{BIX}(u,v) = \begin{cases} v + \psi(v)\log(u), & \textit{if} \quad v \leq u(1 - \log(u)) \\ u, & \textit{if} \quad v > u(1 - \log(u)) \end{cases} \tag{3.8}$$

Proof.
(i) First note that $z_2 \leq z_1 \Rightarrow \Upsilon(z_2) \geq \Upsilon(z_1)$ which implies $u(1-\log(\psi(v))) \leq u(1-\log(u))$. Since $\psi(v) \leq u$, and $v = \psi(v)(1 - \log(\psi(v)))$, $v \leq u(1 - \log(u))$. Thus,

for $v \le u(1-\log(u))$, $G(G_1^{-1}(u), G_2^{-1}(v)) = \exp\{-\frac{v-\psi(v)}{\psi(v)}\}(1+\frac{v+\psi(v)}{\psi(v)}+\log(u))$ $= \exp\{-\frac{v-\psi(v)}{\psi(v)}\}(\frac{v}{\psi(v)}+\log(u))$. From $v = \psi(v)(1-\log(\psi(v)))$ we obtain $G(G_1^{-1}(u), G_2^{-1}(v)) = \psi(v)(1-\log(\psi(v))) + \psi(v)\log(u)$ and the result follows.
(ii) When $z_1 < z_2$, and since, for $\xi > 0$, $\Upsilon(z)$ is a decreasing function of z, we have: $\exp\{-\Upsilon(z_1)\} = G_1(z_1) = u < \psi(v) = G_1(z_2) < G_2(z_2) = v$, and thus $C_{BIX}(u,v) = u = \min(u,v)\Box$

Remark 3.3. The bi-extremal copula $C_{BIX}(u,v)$ is non-exchangeable.

To help understanding the bi-extremal copula, Figure 2 shows in the upper row the contours of the C_{BIX} (at left) and the contours of the density function c_{BIX} (at right). The bottom part of the figure shows the diagonal section of the copulas C_{BIX}, M, and I (at left), and the diagonal section of the corresponding densities (at right).

Theoretical examples

We now provide two examples where the C_{BIX} can be obtained analytically. Let $X_1, X_2, ...X_n$ be a sequence of iid random variables with common distribution F and density f. Assume all notations given. Then, from (2.3) and (2.4) we obtain $F_{1,n}(y) = (F(y))^n$, $F_{2,n}(x) = (F(x))^n + n(F(x))^{n-1}(1-F(x))$, with densities $f_{1,n}(y) = n(F(y))^{n-1}f(y)$, $f_{2,n}(x) = n(n-1)f(x)(F(x))^{n-2}(1-F(x))$, for $x < y$. The joint cdf $F_{12,n}(y,x)$ is given by

$$F_{12,n}(y,x) = n(F(x))^{n-1}(F(y)-F(x))$$

and the joint density is

$$f_{12,n}(y,x) = n(n-1)\left(F(x)\right)^{n-2} f(y)f(x), \quad \text{for} \;\; x < y.$$

Let a_n and b_n be the normalizing constants. The standardized order statistics are $M_i^* = (M_{i,n} - b_n)/a_n$, $i = 1,2$, with distributions $F_{M_i^*}(z_i) = F_{i,n}(a_n z_i + b_n)$, density functions $f_{M_i^*}(z_i) = a_n f_{i,n}(a_n z_i + b_n)$, $i = 1,2$, joint cdf $F_{M_1^*,M_2^*}(z_1,z_2) = F_{12,n}(a_n z_1 + b_n, a_n z_2 + b_n)$, and joint density function $f_{M_1^*,M_2^*}(z_1,z_2) = (a_n)^2 f_{12,n}(a_n z_1 + b_n, a_n z_2 + b_n)$.
(i) *The* $Uniform(0,1)$ *case.* Let F be the $Uniform(0,1)$. Then, $F_{1,n}(y) = y^n$, $F_{2,n}(x) = nx^{n-1}(1-x) + x^n$, $f_{1,n}(y) = ny^{n-1}$, $f_{2,n}(x) = n(n-1)x^{n-2}(1-x)$, for $0 < x < y < 1$. The joint cdf is given by

$$F_{12,n}(y,x) = nx^{n-1}(y-x)\ ,$$

and the joint density is

$$f_{12,n}(y,x) = n(n-1)x^{n-2}, \quad \text{for} \;\; 0 \le x < y \le 1.$$

Let $a_n = \frac{1}{n}$ and $b_n = 1$. Then

$$f_{M_1^*}(z_1) = (1+\frac{z_1}{n})^{n-1}$$

and

$$f_{M_2^*}(z_2) = (\frac{n-1}{n})(-z_2)(1+\frac{z_2}{n})^{n-2}.$$

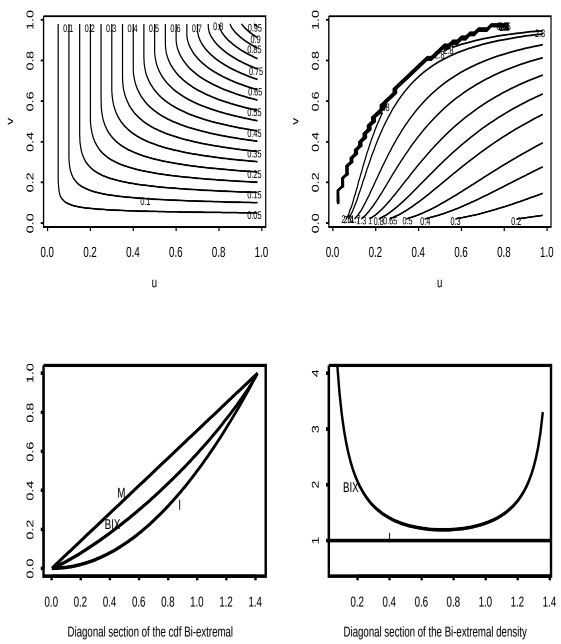

Figure 2 *The upper part of the figure shows the contours of the bi-extremal copula C_{BIX} (at left) and of the density function c_{BIX} (at right). The bottom part of the figure shows at the left hand side the main diagonal section of the copulas C_{BIX}, M (positive perfect depen dence),and I (product); and at the right hand side the diagonal sectionsof the corresponding densities.*

The joint cdf is

$$F_{M_1^*,M_2^*}(z_1,z_2) = n\left(\frac{z_2}{n}+1\right)(z_1-z_2)\ ,$$

and the density becomes

$$f_{M_1^*,M_2^*}(z_1,z_2) = \left(\frac{n-1}{n}\right)\left(\frac{z_2}{n}+1\right)^{n-2}$$

Now, plugging the expressions above in (2.2), we obtain

$$c(F_{M_1^*}(z_1), F_{M_2^*}(z_2)) = \frac{1}{(-z_2)(1+\frac{z_1}{n})^{n-1}} \longrightarrow_{n\to\infty} \frac{-1}{z_2\exp\{z_1\}} .$$

Now, in the Weibull case with $\xi = -1$ we have $G_1(z) = \exp\{z\}$. Since $u = G_1(z_1)$ and $\psi(v) = G_1(z_2)$, we obtain $c(u,v) = \frac{-1}{u\log(\psi(v))}$ □

Remark 3.4. The copula C_{BIX} may be obtained taking the limit as $n \to \infty$ of the right hand side of (3.7) based on the expressions and normalizing constants given above.

(i) *The Exponential*(1) *case.* Let F be the *Exponential*(1). Then, $f_{1,n}(y) = n(1-\exp\{-y\})^{n-1}\exp\{-y\}$, $f_{2,n}(x) = n(n-1)\exp\{-2x\}(1-\exp\{-x\})^{n-2}$, for $0 < x < y$. The joint density is

$$f_{12,n}(y,x) = n(n-1)(1-\exp\{-x\})^{n-2}\exp\{-x\}\exp\{-y\}, \quad \text{for} \ \ 0 \le x < y.$$

Let $a_n = 1$ and $b_n = \log(n)$. Then

$$f_{M_1^*}(z_1) = \exp\{-z_1\}\left(1 - \frac{\exp\{-z_1\}}{n}\right)^{n-1}$$

and

$$f_{M_2^*}(z_2) = \frac{n-1}{n}\exp\{-2z_2\}\left(1 - \frac{\exp\{-z_2\}}{n}\right)^{n-2}.$$

The joint density function is given by

$$f_{M_1^*,M_2^*}(z_1,z_2) = \frac{n-1}{n}\exp\{-z_1\}\exp\{-z_2\}\left(1 - \frac{\exp\{-z_2\}}{n}\right)^{n-2}.$$

Now, plugging the expressions above in (2.2), we obtain

$$c(F_{M_1^*}(z_1), F_{M_2^*}(z_2)) = \lim_{n\to\infty} \frac{1}{\exp\{-z_2\}\left(1 - \frac{\exp\{-z_1\}}{n}\right)^{n-1}} \tag{3.9}$$

$$= \frac{-1}{-\exp\{-z_2\}\exp\{-\exp\{-z_1\}\}}. \tag{3.10}$$

Now, in the Gumbel case, $\xi = 0$ implies $G_1(z) = \exp\{-\exp\{-z\}\}$. Since $u = G_1(z_1)$ and $\psi(v) = G_1(z_2)$, we obtain $c(u,v) = \frac{-1}{u\log(\psi(v))}$ □

Examples using selected *iid* processes

Aiming to assess the adequacy of the asymptotic results for finite samples, we carried out 8 simulation experiments designed as follows. First, N iid observations from a selected process were generated. Then, the series was divided in blocks of size n and the two largest order statistics were collected. The bivariate extreme value distribution (2.8) was fitted to the pairs of observations, and the parameters estimates used in (2.5) to obtain the pseudo-Uniform$(0, 1)$ observations. To check if the data could be properly modeled by the bi-extremal copula, we carried on a goodness of fit test based on a simple direct chi-square approach described in Patton (2001) (see also Fermanian, 2005), a bivariate extention of the usual Pearson test[1]. We report the mean and the standard error from 200 repetitions of each experiment.

Table 1: *P-values means and standard errors from simulations.*

	N(0,1)	N(0,4)	t-st(3)	t-st(6)
Small sample size (30)				
mean	0.4881	0.4767	0.4950	0.4781
stand. error	0.3840	0.3678	0.3742	0.3387
Moderate sample size (100)				
mean	0.4966	0.4966	0.4918	0.4542
stand. error	0.3427	0.3427	0.3460	0.4135

The samples simulated are from the following distributions: (*i*) Normal(0,1), (*ii*) Normal(0,4), (*iii*) standard t-student (3 d.f.), and (*iv*) standard t-student (6 d.f.). We set the block size $n = 30$ and, for each process, we considered two series length, $N = 900$ and $N = 3000$, thus collecting 30 and 100 observations of the two largest order statistics, playing the role of small and moderate data sets.

The results are summarized in Table 1. As expected, the p-values means are close to 0.50, accepting the null hypothesis that the data may be well modeled by the bi-extremal copula.

Dependence measures

We now derive the values of some well known dependence measures, which summarize the dependence structure captured by the bi-extremal copula.

Proposition 3.1. *The tail dependence coefficient : The lower and the upper tail dependence coefficients of the bi-extremal copula are equal to zero.*

Proof. Let $\delta(t)$ represent the diagonal section of a copula, $C(t, t)$. The tail dependence coefficients λ_U and λ_L of a copula C may be computed using $\lambda_U = 2 - \lim_{t\to 1} \partial\delta(t)/\partial(t)$, and $\lambda_L = \lim_{t\to 0} \partial\delta(t)/\partial(t)$ (Nelsen, 2006). In the case of the bi-extremal copula, first note the pairs (t, t) lie bellow the constraint.

[1]We divide the unit square in $k \times k$ squares. Let $O_{i,j}$ and $E_{i,j}$, $i, j = 1, .., k$, be the number of observed and expected frequencies in each (i, j) cell. The test statistic is $\sum_i^k \sum_j^k \frac{(O_{i,j}-E_{i,j})^2}{E_{i,j}}$, which follows a chi-square distribution with ν degrees of freedom. Clearly the power of the test depends on the choice of cells as in any chi-square goodness-of-fit test. We chose a uniform grid on the $[0, 1]^2$, resulting in 1600 squares, and since under the true model the denominator in the test statistic is zero for many cells, we set ν as the number of cells with positive probability mass minus 1.

Thus, $\delta(t) = C_{BIX}(t,t) = t+\bar{t}\ln t$, where $\bar{t}$ satisfies $t = \bar{t}(1-\ln \bar{t})$. Now, $\frac{\partial t}{\partial \bar{t}} = -\ln \bar{t}$ $\Longrightarrow \frac{\partial \bar{t}}{\partial t} = -\frac{1}{\ln \bar{t}}$. Then $\partial\delta(t)/\partial(t) \equiv \delta'(t) = 1 + \frac{\bar{t}}{t} - \frac{\ln t}{\ln \bar{t}}$, and :
(i) $\lim_{t\to 1} \delta'(t) = \lim_{t\to 1}(1 + \frac{\bar{t}}{t} - \frac{\ln t}{\ln \bar{t}}) = 2 - \lim_{t\to 1} \frac{\ln t}{\ln \bar{t}}$. By applying the L'Hopital rule we obtain $\lim_{t\to 1} \delta'(t) = 2 + \lim_{t\to 1} \frac{\bar{t}\ln \bar{t}}{t} = 2$, which implies $\lambda_U = 0$.
(ii) $\lim_{t\to 0} \delta'(t) = 1 + \lim_{t\to 0} \frac{\bar{t}}{t} - \lim_{t\to 0} \frac{\ln t}{\ln \bar{t}}$. By applying the L'Hopital rule we obtain $\lim_{t\to 0} \delta'(t) = 1 - \lim_{t\to 0} \frac{1}{\ln \bar{t}} - \lim_{t\to 0} \frac{\bar{t}\ln \bar{t}}{t} = 1 + \lim_{t\to 0} \frac{\bar{t}\ln \bar{t}}{t}$, and again, by the L'Hopital rule, $\lim_{t\to 0} \delta'(t) = 1 - \lim_{t\to 0} 1 - \lim_{t\to 0} \frac{1}{\ln \bar{t}}$, which implies $\lambda_L = 0$.

Proposition 3.2. *The Spearman's ρ: The Spearman's correlation coefficient, $\rho_{S,BIX}$, of the bi-extremal copula is equal to* $2/3$.

Proof. The definition of $\rho_{S,BIX}$ is (Nelsen, 2006)

$$\rho_{S,BIX} = 12 \int\int_{\mathbf{I}^2} C_{BIX}(u,v)dudv - 3$$

Two cases should be considered

$$\int\int_{\mathbf{I}^2} C_{BIX}(u,v)dudv = \int_0^1 \int_0^{u(1-\log(u))} (v+v_1\log(u))dudv + \int_0^1 \int_{u(1-\log(u))}^1 ududv\,.$$

Now, $\int_0^1 \int_{u(1-\log(u))}^1 ududv = \int_0^1 u[1-u(1-\log((u))]du = \frac{1}{18}$. Changing variables, $v = v_1(1-\log(v_1))$, $dv = -\log(v_1)dv_1$, we obtain $\int_0^1 \int_0^u (v_1 - v_1\log(v_1) + v_1\log(u))(-\log(v_1))dv_1 = \frac{1}{4}$. Thus, $\int\int_{I^2} C_{BIX} = \frac{11}{36}$, and, finally, $\rho_{S,BIX} = \frac{2}{3}$.

We recall that the Spearman's ρ of a pair of continuous rv is identical to the Pearson's linear correlation coefficient between their *grades* (their probability integral transformations).

Proposition 3.3. *The Kendall's tau: The Kendall's correlation coefficient, τ_{BIX}, of the bi-extremal copula is equal to* $1/2$.

Proof. From Theorem 5.1.3, Nelsen (2006), τ_{BIX} is given by

$$\tau_{BIX} = 4\int\int_{\mathbf{I}^2} C_{BIX}(u,v)dC_{BIX}(u,v) - 1.$$

Now,

$$\int\int_{\mathbf{I}^2} C_{BIX}(u,v)dC_{BIX}(u,v) = \int_0^1 \int_0^{u(1-\log(u))} \frac{-1}{u\log(\psi(v))}(v+\psi(v)\log(u))dudv$$

$$= \int_0^1 \int_0^{u(1-\log(u))} \left(\frac{-\psi(v)}{u\log(\psi(v))} + \frac{\psi(v)}{u} - \frac{\psi(v)\log(u)}{u\log(\psi(v))}\right) dudv.$$

By changing variables we get

$$\int_0^1 \int_0^{u(1-\log(u))} \frac{-\psi(v)}{u\log(\psi(v))} dudv = \int_0^1 \frac{1}{u} du \int_0^u \frac{-\psi(v)}{\log(\psi(v))}(-\log(\psi(v)))d\psi(v) = \frac{1}{4}.$$

Also

$$\int_0^1 \int_0^{u(1-\log(u))} \frac{\psi(v)}{u} dudv = \int_0^1 \frac{1}{u} du \int_0^u \psi(v)(-\log(\psi(v)))d\psi(v)$$

$$= \int_0^1 \frac{-1}{u} du \int_0^u \psi(v)\log(\psi(v))d\psi(v) = \int_0^1 \frac{-1}{u}\left(\frac{u^2\log(u)}{2} - \frac{u^2}{4}\right) du$$

$$= (\frac{-1}{2})(\int_0^1 u\log(u)du) + \frac{1}{8} = (\frac{-1}{2})(\frac{-1}{4}) + (\frac{1}{8}) = \frac{2}{8},$$

and

$$\int_0^1 \int_0^{u(1-\log(u))} \frac{\log(u)}{u} \frac{\psi(v)}{\log(\psi(v))} dudv = \int_0^1 \frac{\log(u)}{u} du \int_0^u -\psi(v)d\psi(v)$$

$$= \int_0^1 \frac{-u\log(u)}{2} du = \frac{1}{8},$$

and the result follows □

Remark 3.5. This result may be obtained as the limit as $n \to \infty$ of the expression $1 - \binom{n}{n-1}^2 / \binom{2n}{2(n-1)}$ derived in *Proposition 9* of Avérous et al. (2005).

Remark 3.6. It follows from *Remark 3.2* that the Kendall's tau correlation coefficient of the limit distribution of the smallest and the second smallest order statistics is also equal to $1/2$.

Remark 3.7. Note $\rho_{S,BIX} > \tau_{BIX}$, as should be according to results (the pair of order statistics is positive likelihood ratio dependent) in Nelsen (1992), Capéràa and Genest (1993), and Boland et al. (1996). Fredricks and Nelsen (2007) obtain the limit $3/2$, as $n \to \infty$ for the ratio ρ/τ between the two less dependent order statistics. Here we obtain the limit $4/3$ for the two most dependent order statistic.

Remark 3.8. The Gini's coefficient, $\gamma_{BIX} = 4\left[\int_0^1 C_{BIX}(u, 1-u)du - \int_0^1 [u - C_{BIX}(u,u)\right] du$, of the bi-extremal copula is approximately equal to $2/5$. This result was obtained by numerical integration.

Proposition 3.5. *The Schweizer and Wolff's coefficient: The Schweizer and Wolff's measure of association, σ_{BIX}, of the bi-extremal copula is equal to* $1/18$.

Proof. The definition of σ_{BIX} is (Nelsen, 2006)

$$\sigma_{BIX} = \int\int_{\mathbf{I}^2} \mid C_{BIX}(u,v) - uv \mid dudv.$$

Now, $C_{BIX}(u,v) \geq uv$, for all $u,v \in [0,1]^2$. Since $\int\int_{I^2} C_{BIX} = 11/36$, and $\int\int uvdudv = 1/4$, $\sigma_{BIX} = 1/18$ □

A simulation algorithm

Once one has a newly proposed copula, it is desirable to be able to generate pseudo-observations from it. The classical simulation algorithm (Frees and Valdez, 1998) draws U uniformly from the interval $[0,1]$, and then generate V from the conditional distribution $\partial C_{BIX}(u,v)/\partial v$. The expressions of $\partial C_{BIX}(u,v)/\partial v$ and its inverse are given in *Proposition 3.6* and *Remark 3.9.*

Proposition 3.6. *Let* (U,V) *be a random pair with distribution function* C_{BIX} *and assume all definitions and notations in Theorem 3.1 and Theorem 3.2. The conditional distribution* $C_v(u \mid v) = Pr\{U \leq u \mid V = v\} = 1 - \frac{\log(u)}{\log(\psi(v))}$, *for all* $0 \leq \psi(v), u, v \leq 1$, *and* $v \leq u(1-\log(u))$.

Proof. First note that when $v > u(1-\log(u))$, $C_{BIX}(u,v) = u$, and then $\partial C_{BIX}(u,v)/\partial v = 0$. Thus, for $v \leq u(1-\log(u))$, $C_v(u \mid v) = Pr\{U \leq u \mid V = v\} = \partial C_{BIX}(u,v)/\partial v = \frac{\partial(v+\psi(v)\log(u))}{\partial v} = 1 + \frac{\partial\psi(v)}{\partial v}\log(u)$ □

Remark 3.9. The inverse function of $C_v(u \mid v)$ is given by $C_v^{-1}(q \mid v) = \exp\{(1-q)\log(\psi(v))\}$.

To generate an observation (u,v) from C_{BIX}, one may proceed as follows:

(i) simulate v from $Uniform(0,1)$;

(ii) numerically find $\psi(v)$ implicitly defined by $v = \psi(v)(1-\log(\psi(v)))$;

(iii) simulate q from $Uniform(0,1)$;

(iv) compute $u = \exp\{(1-q)\log(\psi(v))\}$.

Meta-bi-extremal distributions

A distribution is said to be meta-bi-extremal if its associated copula is the bi-extremal. From (2.1), meta-bi-extremal distributions may be constructed by combining (3.8) and any other set of univariate distribution functions. Another examples of meta-distributions may be found in Embrechts et al. (2001), or Fang & Fang (2002), or Demarta and McNeil (2005).

An application

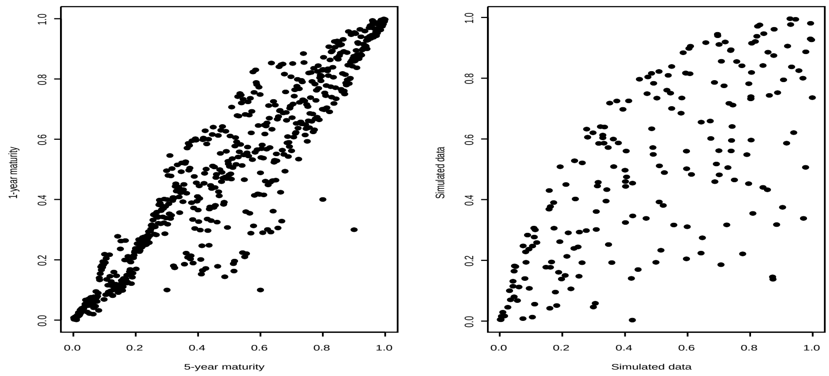

Figure 3 *Scatter plot of Uniform(0,1) data: (left) monthly U.S. interest rates; (right) data simulated from the bi-extremal copula.*

U.S. monthly interest rates. The dependence structure of the Bi-extremal copula may also represent the association between series of interest rates with different maturities. We use the U.S. monthly interest rates obtained from the Federal Reserve Bank of St. Louis. They are the 5-year and the 1-year Treasury constant maturity rates from April 1954 to January 2001. As pointed out by Tsay (2002) they moved in unison. We now model the margins non-parametrically and use the empirical distribution function to obtain the pseudo Uniform$(0,1)$ data. To check the adequacy of the bi-extremal copula for these data we carry on the goodness of fit test described in Footnote 2, which provided a p-value of 0.42, confirming the good adherence of the bi-extremal copula to the data.

4 Further remarks

There are at least two further issues related to the bi-extremal copula which we would like to develop in a further work. The first one is a parametric approximation of the bi-extremal copula, to better adapt to real data. We would like to have an alternative model that can play the role of the bi-extremal copula and, furthermore, adding more flexibility through the introduction of a parameter. A possibility is to replace $\psi(v)$ by v^δ in the expression of the copula C_{BIX}, where $\delta \geq 1$. The second issue is on the generalization of the bi-extremal copula to higher dimensions, the r-extremal copula, $r \geq 3$. We already have a recurrence formula whose some proof passages need to be clarified.

Besides environment and finance, many other situations may be modeled by the bi-extremal copula. They include accounting data from actuary, for example, data on claims and administrative expenses, and data from medical studies, where many patients measurements present strong dependence.

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