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THE GENERALIZED HYPERBOLIC PROCESS

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ABSTRACT. A Stationary non-Gaussian random process called the generalized hyperbolic process is introduced. The process generates a family of processes. We derived the rate of up-crossings for this process as well as the distribution of the height of the process. We proposed an algorithm to simulate the generalized hyperbolic process.

1. INTRODUCTION

For a differentiable random process $X(t),\ t \in [0, A], A > 0$, an up-crossing of the level u occurs at $t_0 \in [0,\ A]$, if $X(t_0) = u$ and $\dot{X}(t_0) > 0$, where $\dot{X}(t)$ refers to the derivative of $X(t)$. Let $N(u, X)$ be the number of up-crossings of u by $X(t)$. The mean number of the random variable $N(u, X)$ is given (under certain conditions) by the Rice's Formulae (Leadbetter and Spaniolo, 2004):

$$E\{N(u, X)\} = A \int_0^\infty f_{X,\dot{X}}(u,\ y)dy,$$

where $f_{X,\dot{X}}(.,.)$ denotes the joint density of $X = X(0)$ and $\dot{X} = \dot{X}(0)$. The mean value of $N(u, X)$ is very important in many application of random processes. It can be used to approximate the tail distribution of $\sup_{t\in[0,A]} X(t)$, for large values of u and A, i.e.,

$$P\{\sup_{t\in[0,A]} X(t) \geq u\} \approx E\{N(u, X)\}.$$

For more information about the use of $E\{N(u, X)\}$ see Leadbetter and Spaniolo (2002) and Rychlik (1993). If $X(t)$ is a differentiable and stationary Gaussian process with $\lambda_2 = Var(\dot{X}(0))$, then

$$E\{N(u, X)\} \quad = \quad \frac{A\sqrt{\lambda_2}}{2\pi} \exp(-\frac{u^2}{2}). \tag{1.1}$$

Gaussian random processes are usually used to model many random quantities arise in different fields such as sea surface elevation, SOI (Southern Oscillation Index), Financial times series, etc (Rychlik, 1993; Baxevani et al., 2005). Unfortunately, the Gaussian process could not capture the skewness in some financial data (Eberlein, 2001). Also when the depth of the water decreases, the sea surface elevation departs from symmetry.

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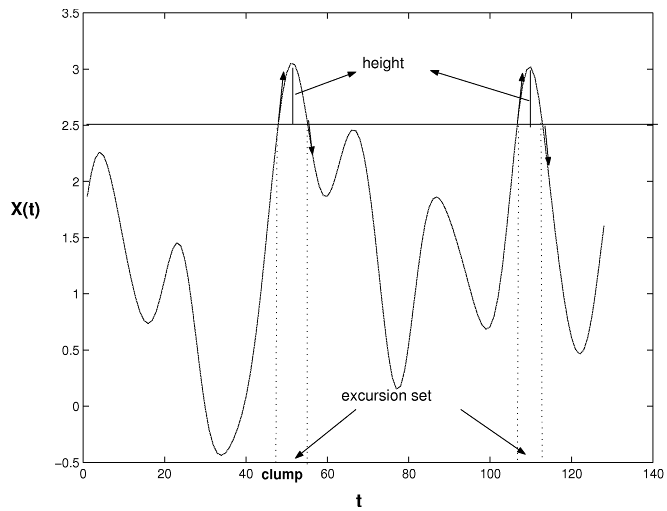

FIGURE 1. Excursion set, up-crossings, down crossings and heights of a random process $X(t)$.

So the Gaussian process is not a suitable model for such data. So we need an alternative model.

In this paper, we introduce a non-Gaussian process. The new process generates a family of processes. We derive the rate of up-crossing for the new process. We also derive the distribution of the height as well as the distribution of the largest height of the process, see Figure 1.

The following notation will be used. $|.|$ for matrix determinant. We use $Z \sim N(\mu, \sigma^2)$ to mean that Z is a normal random variable with mean μ and variance σ^2. The density and cumulative distribution function of the standard normal random variable are $\phi(.)$ and $\Phi(.)$, respectively. If $Z \sim N(\mu, \sigma^2)$ and $Z^+ = \max\{0, Z\}$, then

$$E\{Z^+\} = \mu(1 - \Phi(-\frac{\mu}{\sigma})) + \frac{\sigma}{\sqrt{2\pi}} \exp(-\frac{\mu^2}{2\sigma^2}). \tag{1.2}$$

A random variable W is said to have a generalized inverse Gaussian distribution if it has the pdf

$$f_W(w; \lambda, \chi, \psi) = \frac{\chi^{-\lambda}(\sqrt{\chi\psi})^{\lambda}}{2K_\lambda(\sqrt{\chi\psi})} w^{\lambda-1} \exp(-\frac{1}{2}(\chi w^{-1} + \psi w)), \quad w > 0, \tag{1.3}$$

where $K_\lambda(.)$ is the modified Bessel function of the third kind with index λ. We will use the notation $W \sim N^-(\lambda, \chi, \psi)$ to mean that the random variable W follows the generalized inverse Gaussian distribution with parameters λ, χ and ψ. The function $K_\lambda(x)$ has the following properties

$$K_\lambda(x) \approx \Gamma(\lambda) 2^{\lambda-1} x^{-\lambda}, \text{ as } x \downarrow 0, \tag{1.4}$$

$$K_\lambda(x) \quad \approx \quad \sqrt{\frac{\pi}{2x}} \exp(-x), \quad \text{as} \quad x \uparrow \infty, \tag{1.5}$$

$$K_\lambda(x) \quad = \quad K_{-\lambda}(x). \tag{1.6}$$

It can be shown that

$$E\{W^\alpha\} \quad = \quad \left(\frac{\chi}{\psi}\right)^{\alpha/2} \frac{K_{\lambda+\alpha}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})} \tag{1.7}$$

Blasild (1981) defines the generalized hyperbolic distribution as follows: A random vector V is said to have a d-dimensional generalized hyperbolic distribution if it has the joint density

$$f_{\mathbf{V}}(\mathbf{v}) \quad = \quad c\frac{K_{\lambda-\frac{d}{2}}\left(\sqrt{(\chi + (\mathbf{v}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{v}-\boldsymbol{\mu}))(\psi + \boldsymbol{\gamma}^T\Sigma^{-1}\boldsymbol{\gamma})}\right)\exp((\mathbf{v}-\boldsymbol{\mu})^T\Sigma^{-1}\boldsymbol{\gamma})}{\left(\sqrt{(\psi + (\mathbf{v}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{v}-\boldsymbol{\mu}))(\psi + \boldsymbol{\gamma}^T\Sigma^{-1}\boldsymbol{\gamma})}\right)^{\frac{d}{2}-\lambda}},$$

where

$$c = \frac{(\sqrt{\chi\psi})^{-\lambda}\psi^\lambda(\psi + \boldsymbol{\gamma}^T\Sigma^{-1}\boldsymbol{\gamma})^{\frac{d}{2}-\lambda}}{(2\pi)^{\frac{d}{2}}|\Sigma|^{\frac{1}{2}}K_\lambda(\sqrt{\chi\psi})}.$$

Any $d-$dimensional generalized hyperbolic distribution has the following stochastic representation

$$\mathbf{V} \quad = \quad \boldsymbol{\mu} + \boldsymbol{\gamma}W + \sqrt{W}\mathbf{Z}, \tag{1.8}$$

where $\boldsymbol{\mu}, \boldsymbol{\gamma}$ are constant vectors and W is a positive scalar random variable independent of $\mathbf{Z}$, a multivariate normal vector with mean vector $\mathbf{0}$ and covariance matrix Σ. The mean of $\mathbf{V}$ is given by

$$E\{\mathbf{V}\} = \boldsymbol{\mu} + \boldsymbol{\gamma}\left(\frac{\chi}{\psi}\right)^{1/2} \frac{K_{\lambda+1}(\sqrt{\chi\psi})}{K_\lambda(\sqrt{\chi\psi})}$$

2. Generalized hyperbolic process

Let $X(t)$, $t \in [0, A]$, be a stationary and differentiable Gaussian random process with mean zero and variance one. We define the generalized hyperbolic random process $Y(t)$ by

$$Y(t) = \mu + \gamma W + \sqrt{W}X(t),$$

where $\gamma, \mu \in R$ and $W \sim N^-(\lambda, \chi, \psi)$. It can be noted that every finite dimensional distribution of the process $Y(t)$ is a generalized hyperbolic distribution.

Since $X(t)$ is stationary then so $Y(t)$. If $\mathbf{R}_X(t)$ is the covariance function of $X(t)$, then the covariance function of $Y(t)$ is

$$\begin{aligned} \mathbf{R}_Y(t) &= E\{Y(t)Y(0)\}, \\ &= E\{(\mu + \gamma W + \sqrt{W}X(t))(\mu + \gamma W + \sqrt{W}X(0))\}, \qquad (2.1) \\ &= \gamma^2\mathrm{Var}(W) + E\{W\}\mathbf{R}_X(t). \qquad (2.2) \end{aligned}$$

The last equation implies that $Y(t)$ is differentiable in the mean square sense and $\dot{Y}(t) = \sqrt{W}\dot{X}(t)$.

Theorem 1.

$$E\{N(u,Y)\} = \frac{A\sqrt{\lambda_2}}{2\pi}\left(\frac{\chi^*}{\chi}\right)^{\frac{\lambda}{2}}\frac{K_\lambda(\sqrt{\chi^*\psi})}{K_\lambda(\sqrt{\chi\psi})}. \tag{2.3}$$

Proof. The random variables $X(0)$ and $X\dot(0)$ are independent and normally distributed such that $X = X(0) \sim N(0,1)$ and $\dot{X} = \dot{X}(0) \sim N(0,\lambda_2)$. Conditional on W, the joint distribution of $(Y,\dot{Y}) = (Y(0),\dot{Y}(0))$ is a bivariate normal with joint density function

$$f_{Y,\dot{Y}}(y,\dot{y}|W=w) = \frac{1}{w\sqrt{\lambda_2}}\phi\left(\frac{y-\mu-\gamma E\{W\}}{\sqrt{w}}\right)\phi\left(\frac{\dot{y}}{\sqrt{w\lambda_2}}\right),$$

where $\lambda_2 = Var(\dot{X}(0))$ and $E\{W\}$ is given by (1.7). Using the total probability law and Fubini's theorem, we get

$$\begin{aligned}
E\{N(u,Y)\} &= A\int_0^\infty \dot{y} f_{Y,\dot{Y}}(u,\dot{y})d\dot{y},\\
&= A\int_0^\infty\int_0^\infty \dot{y} f_{Y,\dot{Y}}(u,\dot{y}|W=w)f_W(w)dwd\dot{y},\\
&= A\int_0^\infty\int_0^\infty \frac{\dot{y}}{w\sqrt{\lambda_2}}\phi\left(\frac{u-\mu-\gamma E\{W\}}{\sqrt{w}}\right)\phi\left(\frac{\dot{y}}{\sqrt{w\lambda_2}}\right)f_W(w)d\dot{y}dw,\\
&= A\int_0^\infty \frac{\dot{y}}{\sqrt{w}}\phi\left(\frac{u-\mu-\gamma E\{W\}}{\sqrt{w}}\right)E\{\dot{Y}^+|W=w\}f_W(w)dw,\\
&= A\int_0^\infty \frac{1}{\sqrt{w}}\phi\left(\frac{u-\mu-\gamma E\{W\}}{\sqrt{w}}\right)\left(\sqrt{\frac{w\lambda_2}{2\pi}}\right)f_W(w)dw,\\
&= \frac{A\sqrt{\lambda_2}}{2\pi}\frac{\chi^{-\lambda}(\sqrt{\chi\psi})^\lambda}{2K_\lambda(\sqrt{\chi\psi})}\int_0^\infty w^{\lambda-1}\exp(-\frac{1}{2}(\frac{\chi^*}{w}+\psi w))dw, \qquad (2.4)
\end{aligned}$$

where $\chi^* = (u-\mu-\gamma E\{W\})^2+\chi$. We note that the integrand in (2.4) is similar to the density of $N^-(\lambda,\chi^*,\psi)$. So normalizing this integrand and using (1.3), we end the proof. □

For large values of u and A, the number of up-crossings, $N(u,Y)$ can be approximated by a Poisson point process with mean $E\{N(u,Y)\}$. We may use the standard parametric methods for the Poisson distribution to estimate the parameters $\lambda,\ \psi,\chi$ and λ_2, if realizations of the point process $N(u,\ X)$ are available.

3. Special cases

In this section we discuss special cases of the generalized hyperbolic process. If S is a chi-square random variable with ν degrees of freedom, then the random variable $W = \nu/S$ has the density

$$f_W(w) = \frac{w^{-\nu/2-1}\exp(-\frac{\nu}{2w})}{\Gamma(\frac{\nu}{2})2^{\frac{\nu}{2}}\nu^{\frac{\nu}{2}}} \quad w>0,$$

which is $N^{-}(-\frac{\nu}{2},\ \nu, \psi)$ as $\psi \to 0$. For the case $\mu = \gamma = 0$, $Y(t)$ is a student random variable with ν degrees of freedom. Therefore,

$$E\{N(u, Y)\} \quad = \quad \frac{A\sqrt{\lambda_2}}{2\pi}\left(\frac{u^2+\nu}{\nu}\right)^{-\frac{\nu}{4}} \frac{K_{-\frac{\nu}{2}}(\sqrt{\chi^*\psi})}{K_{-\frac{\nu}{2}}(\sqrt{\chi\psi})}.$$

Using (1.4)-(1.6), it is easy to show that

$$\lim_{\psi\to 0} \frac{K_{-\frac{\nu}{2}}(\sqrt{\chi^*\psi})}{K_{-\frac{\nu}{2}}(\sqrt{\chi\psi})} \quad = \quad \left(1+\frac{u^2}{\nu}\right)^{-\frac{\nu}{4}}.$$

This leads to

$$E\{N(u, Y)\} \quad = \quad \frac{A\sqrt{\lambda_2}}{2\pi}\left(1+\frac{u^2}{\nu}\right)^{-\frac{\nu}{2}}, \tag{3.1}$$

which is the rate of the up-crossings of the student process introduced in Alodat and Aludaat (2007). As $\nu \to \infty$, the student distribution goes to the Gaussian distribution. Note that the right hand side of (3.1) tends to $E\{N(u, X)\}$, the expected number of up-crossings of u by the Gaussian process $X(t)$.

4. Height distribution of $Y(t)$

The distribution of the height of the process $Y(t)$ is defined as the distribution of $Y(0) - u$ given that $Y(t)$ has a local maximum of height exceeds u at $t = 0$. We give the following theorem about $G(y)$, the height distribution of $Y(t)$.

Theorem 2.

$$G(y) \quad = \quad 1 - \exp(-\sqrt{\psi}y), \quad y > 0; \;\; \psi > 0.$$

Proof. As $u \to \infty$, the ratio $\frac{E\{N(u+y,Y)\}}{E\{N(u,Y)\}}$ tends to $1 - G(y)$, where $G(y)$ is the distribution of the height of the process $Y(t)$ above the level u, See Rychlik (1993). Using (1.5) and taking this limits as $u \to \infty$, we get

$$\begin{aligned} 1 - G(y) \quad &= \quad \lim_{u\to\infty} \exp\left(-\sqrt{((u+y-\mu-\gamma E\{W\})^2+\chi)\psi} + \sqrt{((u-\mu-\gamma E\{W\})^2+\chi)\psi}\right), \\ &= \quad \exp(-\sqrt{\psi}y), \quad y > 0; \;\; \psi > 0. \end{aligned}$$

□

Leadbetter and Spaniolo (2002) discussed other statistical properties for the stationary processes such as the distribution of the process after an up-crossing. Such properties can be derived in closed forms for the generalized hyperbolic process.

5. Largest height distribution

Aldous (1989) introduced the concept of Poisson Clumping Heuristic (PCH) which means throwing random sets in the space according to a Poisson point process. The sets are called clumps and the Poisson points are called the centers. He used the PCH to model the excursion set of a random process $X(t)$, i.e., the set $\{t \in [0, A] : X(t) \geq u\}$. The clumps sizes and height are approximately independent. Let M denote the number of clumps in the excursion of $Y(t)$ and $Y_1, \ldots, Y_M$ denote the heights of the local maxima of $Y(t)$. Then M has a Poisson distribution with mean $\alpha = E\{N(u, Y)\}$. According to Sec. 4, $Y_1, Y_2, \ldots, Y_M$ is a random sample from $G(y)$. We are interested in finding the distribution of the largest height of the process, i.e., the distribution of the random variable $V = \max_{i=1}^{M} Y_i$. So

$$\begin{aligned} P\{V \leq v | M \geq 1\} &= P\{\max_{i=1}^{M} Y_i | M \geq 1\} = \sum_{m=1}^{\infty} \frac{P\{Y_1 \leq v, \ldots, Y_m \leq v, M = m\}}{P\{M \geq 1\}}, \\ &= \sum_{m=1}^{\infty} P\{Y_1 \leq v\}^m \frac{\exp(-\alpha\})\alpha^m}{m! P\{M \geq 1\}}, \\ &= \frac{\exp(-\alpha)(\exp(\alpha G(v)) - 1)}{1 - \exp(-\alpha)}. \end{aligned}$$

The probability $P\{V > v\}$ could be very important in application of random processes to Oceanography, since it represents a reliability measure for the sea state above a given threshold v.

6. Simulation

To simulate the generalized hyperbolic process $Y(t)$ we propose the follow algorithm

(1) Simulate a generalized inverse Gaussian variable W.
(2) Simulate a Gaussian process $X(t)$.
(3) Realize $Y(t)$ as $Y(t) = \mu + \gamma W + \sqrt{W} X(t)$.

Accept-reject method can be used to simulate samples form $f_W(w)$. The accept-reject method is given in Rubinstein (1981). This method can be described as follows. If $f_W(w) = cg(w)h(w)$, where c is a normalizing constant, $0 < g(w) \leq 1$ and $h(w)$ is a density function, then

(1) Simulate U from un uniform(0,1).
(2) Simulate W from $h(w)$.
(3) If $U < g(W)$, then deliver W from $f_W(w)$.

For the GIG distribution we choose

$$c = \frac{\chi^{-\lambda}(\sqrt{\chi\psi})^{\lambda}\Gamma(\lambda)(\frac{2}{\psi})^{\lambda}}{2K_{\lambda}(\sqrt{\chi\psi})},$$

$$g(w) = \exp(-\frac{\psi w}{2})$$

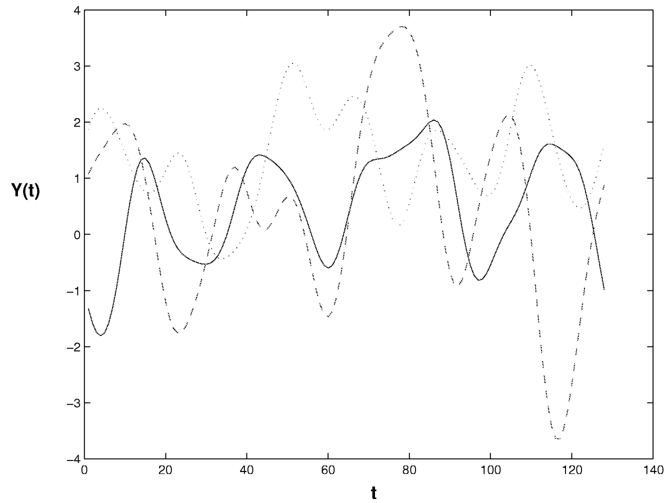

FIGURE 2. Simulation of the generalized hyperbolic processes in the interval $[0, 128]$ for different values of χ, ψ and λ: Solid ($\chi = 1,\ \psi = 1$ and $\lambda = 2$), dotted ($\chi = 2,\ \psi = 1$ and $\lambda = 1$) and dashed ($\chi = 1,\ \psi = 2$ and $\lambda = 1$).

and

$$h(w) = \frac{w^{\lambda-1}\exp(-\frac{\psi}{2}w)}{\Gamma(\lambda)(\frac{2}{\psi})^{\lambda}}.$$

The simulation of the Gaussian process $X(t)$ can be obtained by convolving a Gaussian white noise with a Gaussian kernel. We use the above algorithm to simulate realizations of $Y(t)$. Figure 2 shows these realizations for different values of χ, ψ and λ.

7. Conclusion

In this paper, we introduced a non-Gaussian process. The process is called the generalized hyperbolic process. The generalized hyperbolic process generates a family of random processes. As a limit, the Gaussian and the student processes are members of this family. This makes the generalized hyperbolic process a flexible model in many practical problems.

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