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A NOTE ON SOME PITFALLS ON THE SAWYER TEST FOR DISCRIMINATING BETWEEN SEPARATE FAMILIES OF HYPOTHESES

FRANCISCO ANTONIO ROJAS-ROJAS, FRANCISCO LOUZADA-NETO,
AND BASILIO DE BRAGANÇA PEREIRA

ABSTRACT. In this paper we examine the Sawyer test for discriminating between lognormal and exponential families and between lognormal and Weibull families. We learn that the Sawyer test can not be used for discriminating these separate families of hypotheses.

1. Introduction

Sawyer (1983) proposed an alternative test to the Cox test (Cox, 1961, 1962) for discriminating between two separate families of hypotheses based on the information criterion. His procedure has been considered in the econometric modelling context. Interested readers can refer to Mizon and Richard (1986) and McAller (1995).

Considering two separate Gaussian linear regression models, Sawyer (1983) pointed out that his test procedure is a feasible alternative to the Cox test. However, in this paper we show that the Sawyer test can not be used for discriminating neither between lognormal and exponential families nor between lognormal and Weibull families. In Section 2 we briefly review the Cox test. In Section 3 we describe the pitfalls pointed out above. Some final discussion in Section 4 concludes the paper.

2. The Cox test

Consider two separate families of hypotheses H_f and H_g related to densities $f(y,\alpha)$ and $g(y,\beta)$, respectively, where α is the vector of the parameters of the f density and β is the vector of the parameters of the g density. The Cox test for testing H_f against H_g employs the statistic (Cox, 1961, 1962)

$$T_f(\hat{\alpha}) = L_{fg}(\hat{\alpha},\hat{\beta}) - E_{\hat{\alpha}} L_{fg}(\hat{\alpha},\hat{\beta}), \tag{2.1}$$

where $L_{fg}(\alpha,\beta) = L_f(\alpha) - L_g(\beta)$ is the logarithm of the likelihood ratio, $\hat{\alpha}$ and $\hat{\beta}$ are the maximum likelihood estimators of α and β under a random sample of size n, and $E_{\hat{\alpha}}$ is the expectation under H_f evaluated at $\alpha = \hat{\alpha}$. Analogously, for testing H_g against H_f the Cox test employs the statistic

$$T_g(\hat{\beta}) = L_{gf}(\hat{\beta},\hat{\alpha}) - E_{\hat{\beta}} L_{gf}(\hat{\beta},\hat{\alpha}). \tag{2.2}$$

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TABLE 1. Possible outcomes of the Cox test.

$T_g(\hat{\beta})$	$T_f(\hat{\alpha})$ Significant negative	Not Significant	Significant positive
Significant negative	Reject both H_f and H_g	Accept H_f	Reject both H_f and H_g
Not Significant	Accept H_g	Accept both H_f and H_g	Possibly accept H_g
Significant positive	Reject both H_f and H_g	Possibly accept H_f	Inadmissible (Reject Both H_f and H_g)

The outcomes and interpretations of the test are given in Table 1. Details about the asymptotic distributions of (1) and (2) can be found in Cox (1961, 1962). Interested readers can refer to Jackson (1968) and Pereira (1978), who derived the Cox statistics for discriminating between lognormal and exponential families, lognormal and gamma families and lognormal and Weibull families.

As an illustration of the Cox test calculations, following Cox (1961), consider H_f with the lognormal density,

$$f(y,\alpha) = (2\pi\alpha_2)^{-1/2} y^{-1} \exp[-\frac{1}{2\alpha_2}(\log y - \alpha_1)^2], \tag{2.3}$$

where $\alpha = (\alpha_1, \alpha_2)$ and H_g with exponential density,

$$g(y,\beta) = \beta^{-1}\exp(-y/\beta). \tag{2.4}$$

Then,

$$L_{fg}(\hat{\alpha},\hat{\beta}) = -(n/2)\log(2\pi\hat{\alpha}_2) - n\hat{\alpha}_1 + (n/2) + n\log\hat{\beta}$$

and

$$E_{\hat{\alpha}} L_{fg}(\hat{\alpha},\hat{\beta}) = -(n/2)\log(2\pi\hat{\alpha}_2) + (n/2)\hat{\alpha}_2 + (n/2),$$

where $\hat{\alpha}_1 = (1/n)\sum \log y_i$, $\hat{\alpha}_2 = (1/n)\sum(\log y_i - \hat{\alpha}_1)^2$ and $\hat{\beta} = (1/n)\sum y_i$. Therefore,

$$T_f(\hat{\alpha}) = n\log\hat{\beta} - n\hat{\alpha}_1 - \frac{n}{2}\hat{\alpha}_2 = n\log(\frac{\hat{\beta}}{\hat{\beta}_\alpha}),$$

where $\hat{\beta}_\alpha = \exp(\hat{\alpha}_1 + \hat{\alpha}_2/2)$ is the estimated value of β_α, which is the limit in probability of $\hat{\beta}$ under H_f.

Considering the reverse procedure, $L_{gf}(\hat{\beta},\hat{\alpha}) = (n/2)\log(2\pi\hat{\alpha}_2) + n\hat{\alpha}_1 - n\log\hat{\beta} - (n/2)$ and $E_{\hat{\beta}} L_{gf}(\hat{\beta},\hat{\alpha}) = (n/2)\log(2\pi\hat{\alpha}_{2\beta}) + n\psi(1) - (n/2)$, and then

$$T_g(\hat{\beta}) = \frac{n}{2}\log(\frac{\hat{\alpha}_2}{\hat{\alpha}_{2\beta}}) + n(\hat{\alpha}_1 - \hat{\alpha}_{1\beta}),$$

where $\hat{\alpha}_{1\beta} = \log\hat{\beta} + \psi(1)$ and $\hat{\alpha}_{2\beta} = \psi'(1)$ are the estimated values of $\alpha_{1\beta}$ and $\alpha_{2\beta}$, which are the limits in probability of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ under the exponential distribution, respectively, and $\psi(.)$ and $\psi'(.)$ are the digamma and trigamma functions, respectively.

3. The Sawyer test

The Sawyer test for discriminating between two separate families of hypotheses, H_f and H_g, according to Section 1, employs the following statistics (Sawyer, 1983)

$$S_f(\hat{\alpha}) = E_{\hat{\beta}} L_{fg}(\hat{\alpha}, \hat{\beta}) - E_{\hat{\alpha}}[E_{\hat{\beta}} L_{fg}(\hat{\alpha}, \hat{\beta})] \tag{3.1}$$

and

$$S_g(\hat{\beta}) = E_{\hat{\alpha}} L_{gf}(\hat{\beta}, \hat{\alpha}) - E_{\hat{\beta}}[E_{\hat{\alpha}} L_{gf}(\hat{\beta}, \hat{\alpha})]. \tag{3.2}$$

The statistic $S_f(\hat{\alpha})$ is used to detect departures from H_f in the presence of a specific alternative H_g, while $S_g(\hat{\beta})$ is used to detect departures from H_g in the presence of a specific alternative H_f. The Sawyer test outcomes are analogous to the Cox test outcomes presented in Table 1, replacing $T_f(\hat{\alpha})$ and $T_g(\hat{\beta})$ by $S_f(\hat{\alpha})$ and $S_g(\hat{\beta})$. Details about the asymptotic distributions of (5) and (6) can be found in Sawyer (1983).

If we consider H_f and H_g related to the densities (3) and (4), respectively, since

$$E_{\hat{\alpha}} L_g(\hat{\beta}) = -n\hat{\alpha}_1 - (n/2)\hat{\alpha}_2 - n,$$
$$E_{\hat{\alpha}} L_f(\hat{\alpha}) = -(n/2)\log(2\pi\hat{\alpha}_2) - n\hat{\alpha}_1 - n/2,$$
$$E_{\hat{\alpha}} L_{gf}(\hat{\beta}, \hat{\alpha}) = (n/2)\log(2\pi\hat{\alpha}_2) - (n/2)\hat{\alpha}_2 - (n/2)$$

and

$$E_{\hat{\beta}}[E_{\hat{\alpha}} L_{gf}(\hat{\beta}, \hat{\alpha})] = (n/2)\log(2\pi\psi'(1)) - (n/2)\psi'(1) - (n/2),$$

the statistics (6) can be rewritten as

$$S_g(\hat{\beta}) = \frac{n}{2}[\psi'(1) - \hat{\alpha}_2 + \log\hat{\alpha}_2 - \log\psi'(1)].$$

Also, since $E_{\hat{\beta}} L_f(\hat{\alpha}) = -(n/2)\log[2\pi\psi'(1)] - n[\psi(1) + \log\hat{\beta}] - (n/2)$, $E_{\hat{\beta}} L_g(\hat{\beta}) = -n\log\hat{\beta} - n$ and $E_{\hat{\beta}} L_{fg}(\hat{\alpha}, \hat{\beta}) = -(n/2)\log[2\pi\psi'(1)] - n\psi(1) + n/2 = E_{\hat{\alpha}}[E_{\hat{\beta}} L_{fg}(\hat{\alpha}, \hat{\beta})]$, (5) is null. Consequently, it is not possible to discriminate between lognormal and exponential families by using the Sawyer test.

Another drawback in Sawyer test is observed if we consider the test of H_f, with log-normal density (3), against H_g, with a Weibull density, given by

$$g(y, \omega) = \left(\frac{\omega_2}{\omega_1}\right)\left(\frac{y}{\omega_1}\right)^{\omega_2 - 1} \exp\left[-\left(\frac{y}{\omega_1}\right)^{\omega_2}\right].$$

In this case, $E_{\hat{\alpha}} L_f(\hat{\alpha}) = -(n/2)\log[2\pi\hat{\alpha}_2] - n\hat{\alpha}_1 - (n/2)$ and $E_{\hat{\alpha}} L_g(\hat{\omega}) = -(n/2)\log\hat{\alpha}_2 - n\hat{\alpha}_1 - 3(n/2)$, where $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are defined above, and $\hat{\omega}_{1\alpha} = \exp[\hat{\alpha}_1 + (1/2)\sqrt{\hat{\alpha}_2}\,]$ and $\hat{\omega}_{2\alpha} = 1/\sqrt{\hat{\alpha}_2}$ (Pereira, 1978). Therefore $E_{\hat{\alpha}} L_{gf}(\hat{\omega}, \hat{\alpha}) = -n + (n/2)\log(2\pi) = E_{\hat{\omega}}[E_{\hat{\alpha}} L_{gf}(\hat{\omega}, \hat{\alpha})]$ and (6) is null. Besides, denoting α_{1w} and α_{2w} as the limits in probability of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ under the Weibull distribution, their estimated values are $\hat{\alpha}_{1\omega} = \psi(1)/\hat{\omega}_2 + \log\hat{\omega}_1$ and $\hat{\alpha}_{2\omega} = \psi'(1)/\hat{\omega}_2^2$, and then $E_{\hat{\omega}} L_f(\hat{\alpha}) = n\log\hat{\omega}_2 - n\log\hat{\omega}_1 - n\psi(1)/\hat{\omega}_2 - (n/2)\log[2\pi\psi'(1)] - n/2$ and $E_{\hat{\omega}} L_g(\hat{\omega}) = n\log\hat{\omega}_2 - n\log\hat{\omega}_1 - n\psi(1)/\hat{\omega}_2 - n\psi(1) - n$, where $\hat{\omega}_2 = \left[(\sum(\log y_i)\, y_i^{\hat{\omega}_2} / \sum y_i^{\hat{\omega}_2}) - (\sum \log y_i/n)\right]^{-1}$ and $\hat{\omega}_1^{\hat{\omega}_2} = \sum y_i^{\hat{\omega}_2}/n$, then $E_{\hat{\omega}} L_{fg}(\hat{\alpha}, \hat{\omega}) = -(n/2)\log[2\pi\psi'(1)] - n\psi(1) + n/2 = E_{\hat{\alpha}}[E_{\hat{\omega}} L_{fg}(\hat{\alpha}, \hat{\omega})]$ and (5) is null. Then, it is not possible to discriminate between lognormal and Weibull families by considering the Sawyer test.

4. Discussion

We examined the Sawyer statistics for discriminating between lognormal and exponential families and between lognormal and Weibull families. The important lesson learned from this work is that effective discrimination between these separate families of hypotheses is not feasible if the Sawyer test is considered.

It is important to note that the general expressions of the variances of the statistics (1) and (2) and (5) and (6) are given in Cox (1962) and Sawyer (1983), respectively.

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References

Cox D.R. (1961), Tests of separate families of hypotheses, 'Proceedings of the 4th Symposium of Berkeley' 6, pp. 105-123.

Cox, D.R. (1962), 'Further results on tests of separate families of hypotheses', *Journal of the Royal Statatistical Society*, B, pp. 406-424.

Jackson, O.A.Y. (1968), 'Some results on tests of separate families of hypotheses', *Biometrika* **55**, pp. 355-363.

McAller, M. (1995), 'The significance of testing empirical non-nested models', *Journal of Econometrics* **67**, pp. 149-171.

Mizon, G.E., Richard, J.F. (1986), 'The encompassing principle and its application to testing non-nested hypotheses', *Econometrica* **54**, pp. 657-678

Pereira, B.B. (1978), 'Empirical comparations of the some tests of separate families of hypotheses', *Metrica* **25**, pp. 219-234.

Sawyer, K.R. (1983), 'Testing separate families of hypotheses: An information criterion', *Journal of the Royal Statatistical Society*, B. pp. 89-99.

Departamento de Estatística, Universidade Federal de São Carlos, CP 676, São Carlos/SP, 13565-906, Brazil

E-mail address, F.A. Rojas-Rojas: `rojas@power.ufscar.br`

Departamento de Estatística, Universidade Federal de São Carlos, CP 676, São Carlos/SP, 13565-906, Brazil

E-mail address, F. Louzada-Neto: `dfln@power.ufscar.br`

COPPE, Universidade Federal do Rio de Janeiro, CP 68507, Rio de Janeiro/RJ, 21941-972, Brazil

E-mail address, B.de B. Pereira: `basilio@hucff.ufrj.br`